Math 741 - Fall 2015

Homework 2 Presentations: Mon 28 Sep. 2015 5pm

1. **Cyclic Groups.** A group G is said to be cyclic if there exists $g \in G$ such that every element in G is of the form g^n for some integer n. If the order of G is a prime p, show that G is cyclic.

2. **Dodecahedron.** Consider the Dodecahedron $\mathcal{D} \subset \mathbb{R}^3$ and recall the natural action of $SO(3)$ on $\mathcal{D}$. Let $D = Stab_{SO(3)}(\mathcal{D})$. Define a natural map $r : D \to S_5$.

3. **Products in Categories.** Suppose $\mathcal{C}$ is a category, with $A_1, A_2 \in Ob(\mathcal{C})$. An object $A_1 \times A_2 \in Ob(\mathcal{C})$, together with morphisms $\pi_1 : A_1 \times A_2 \to A_1$ and $\pi_2 : A_1 \times A_2 \to A_2$, is called a product of A_1 and A_2 if it satisfies the following universal property:

   (**Product**) Whenever X is an object of $\mathcal{C}$ with morphisms $f_1 : X \to A_1$ and $f_2 : X \to A_2$ then there exists a unique morphism $\phi : X \to A_1 \times A_2$ such that $f_1 = \pi_1 \circ \phi$ and $f_2 = \pi_2 \circ \phi$.

   a. Show that if a product of A_1 and A_2 exists, it must be unique up to unique isomorphism. Namely, if (P, π_1, π_2) and (P', π'_1, π'_2) are both products of A_1 and A_2 then there is a unique isomorphism $\iota : P \to P'$ such that $\pi'_1 = \pi_1 \circ \iota$ and $\pi'_2 = \pi_2 \circ \iota$.

   b. Let Set denote the category of Sets where objects are sets and morphisms are functions. For $X, Y \in Ob(Set)$, explicitly construct the product object $X \times Y$ and morphisms π_1, π_2, and show that it is indeed a product (i.e. show that it satisfies the universal property above).

   c. Let *Vect* be the category whose objects are vector spaces over a field k and whose morphisms are k-linear maps. For $V, W \in Ob(Vect)$, explicitly construct a product object $V \times W$ and show that it is indeed a product.

4. **Sign Homomorphism.** Let $R = \mathbb{Z}[x_1, \ldots, x_n]$ be the set of polynomials in n variables with coefficients in $\mathbb{Z}$.

a. Describe the natural action $\bullet$ of S_n on R.

b. The discriminant is the element

$$D = \prod_{1 \leq i < j \leq n} (x_i - x_j) \in R$$

Show that for any $\sigma \in S_n$, we have $\sigma \bullet D = sign(\sigma)D$ with $sign(\sigma) \in \{\pm 1\}$. We call $sign(\sigma)$ the sign of σ.

c. Show that the map $sgn : S_n \to \{\pm 1\}$ sending $\sigma \mapsto sign(\sigma)$ is a homomorphism.

5. **Alternating Group.** Recall that an element $\tau \in S_n$ is called a transposition if it interchanges two elements and leaves the rest fixed.

a. Show that every $\sigma \in S_n$ can be decomposed as a product of transpositions (i.e. $\sigma = \tau_1 \circ \ldots \circ \tau_m$ where each τ_i is a transposition).

b. Show that if $\sigma = \tau_1 \circ \ldots \circ \tau_m$ and $\sigma = \tau'_1 \circ \ldots \circ \tau'_{m'}$ are two decompositions of σ into a product of transpositions, then $m - m'$ is even.

c. The kernel of the map sgn defined in problem 4c is called the alternating group and is denoted A_n. Consider the subset of S_n consisting of elements which can be decomposed into an even number of transpositions. Show that this subset is equal to A_n.

6. **Quotient of Sets.** Let Y be a subset of X, let $\alpha : X \times Y \to X$ be a map, and $\pi : X \times Y \to X$ be the map defined by $(x, y) \mapsto x$. We define a quotient of X by Y under α to be a set, denoted Q, along with a surjective map of sets $pr : X \to Q$ such that $pr \circ \alpha = pr \circ \pi$, which satisfies that following universal property:

(**Quotient**) Given any surjective map of sets $f : X \to Q'$ such that $f \circ \alpha = f \circ \pi$, there exists a unique surjective map $f' : Q \to Q'$ such that $f = f' \circ pr$.

a. Prove that if a quotient exists, it is unique up to unique isomorphism. Namely, if (Q, pr) and (Q', pr') are both quotients of X by Y under α, there exists a unique isomorphism $\iota : Q \to Q'$ such that $\iota \circ pr = pr'$.

b. Let G be a group and $H < G$ a subgroup.

i. Define G/H and $pr : G \to G/H$ as in class.

ii. Let $\alpha : G \times H \to H$ be the map defined by $\alpha(g, h) = gh$. Show that G/H, along with pr, is a quotient of G by H under α.

Good Luck!