Homework 11. Presentations Mon Dec. 14. 2015 5pm.

Construction of the irreducible representations of S_4. In this HW $G = S_4$, the group of permutations on the set $\{1, 2, 3, 4\}$. In addition, all vector spaces and representations are of finite dimension over $\mathbb{C}$. Finally, we denote by $Irr(G)$ the set of equivalence classes of irreducible representations of G. Our goal is to compute a representative for each class in $Irr(G)$.

1. Compute $\#Irr(G)$.

2. Write down two 1-dimensional representations of G.

3. Let $X = \{1, 2, 3, 4\}$. Decompose the space $\mathcal{H} = \mathbb{C}(X)$ of complex valued functions on X into a direct sum of irreducible representations of G.

4. Let ρ be a non-trivial irreducible representation appearing in problem 3, and define $\pi = sgn \otimes \rho$, where sgn denotes the sign representation of G. Show that $\pi \ncong \rho$.
   Hint: Recall that if V, W are vector spaces, and T, S, are linear operators on V, W, respectively, then we get a map $T \otimes S : V \otimes W \to V \otimes W$ with the property that $tr(T \otimes S) = tr(T) \cdot tr(S)$. Let $\tau \in S_4$ be a transposition, and compute $tr(\pi(\tau))$ and $tr(\rho(\tau))$. It will be of advantage for you to use the following result (prove it if you use it): Let X be a finite G set and consider the permutation representation π_X of G on $\mathbb{C}(X)$. Then $tr(\pi_X(g)) = \#X^g$, where $X^g = \{x \in X; gx = x\}$.

5. **The missing representations.**

   a. Using problem 1., deduce that you are still missing some irreducible representations of G.

   b. Use the formula $\sum_{\rho \in Irr(G)} dim(\rho)^2 = \#G$ to find the dimensions of the irreducible representations you are missing.

   c. Let $\mathcal{C}$ be the cube, and let X denote the set of pairs of antipodal faces of $\mathcal{C}$. Note that G acts on $\mathcal{C}$, and hence on X, which gives rise to the permutation representation $(\pi_X, G, \mathbb{C}(X))$. Decompose π_X into irreducible representations.

d. Using part c, find the remaining irreducible representations of G. To conclude, write a representative for each class in $Irr(G)$.

Good Luck!