Math 741 - Fall 2015

Homework 10. Presentations Mon Dec. 7. 2015 5pm.

1. **Complete Reducibility.** Let us denote by $Irr(G)$ the set of equivalence classes of irreducible representations of a group G on finite dimension vector spaces over a field $\mathbb{F}$. Let (π, G, V) be a representation of G on finite dimensional vector space V over $\mathbb{F}$. Show that the following are equivalent:

   a. $\pi \cong \bigoplus_{\rho \in Irr(G)} m_\rho \cdot \rho$.

   b. For every sub-representation $\tau < \pi$, there exists a sub-representation τ' such that $\pi \cong \tau \oplus \tau'$.

2. **Decomposition into isotypic components is unique.**

   a. Using the trivial action of a group G on an n-dimensional vector space V, show that the decomposition of V into irreducible sub-representations is not unique. (That is, there are many different decompositions of V into a direct sum of irreducible components).

   b. Let (π, G, V) be a representation on a finite dimensional vector space V. Consider a decomposition of V as a direct sum of irreducibles, $V = \bigoplus_{\rho \in Irr(G)} V_\rho$, where each V_ρ is a maximal G-invariant subspace with respect to the property that all of its irreducible sub-representations isomorphic to ρ. Here maximal means that there is no subspace V'_ρ containing V_ρ such that all the irreducible sub-representations of V'_ρ are isomorphic to ρ. Show that if $V = \bigoplus_{\rho \in Irr(G)} W_\rho$ is a second such decomposition of V, then $V_\rho = W_\rho$ for all $\rho \in Irr(G)$. The subspace V_ρ is called the ρ-isotypic component of π.

3. **Intertwining Number.** Recall from homework 9 that we defined the intertwining number $< \pi, \rho >= dim(Hom(\pi, \rho))$ of two representations (π, G, V) and (ρ, G, W), where V and W are finite dimensional vector spaces over $\mathbb{C}$. Recall that the intertwining numbers are symmetric and bilinear, i.e., $< \pi, \rho >=< \rho, \pi >$, and if (τ, G, U) is another representation of G then $< \pi \oplus \tau, \rho >=< \pi, \rho > + < \tau, \rho >$. Also recall that

they satisfy Schur's lemma, i.e., if π, ρ are irreducible, then $< \pi, \rho >= 1$ if $\pi \cong \rho$ and 0 otherwise. Prove the following statements using the intertwining number

a. If $\pi = \bigoplus_{\sigma \in Irr(G)} m_\sigma \cdot \sigma$ and $\pi = \bigoplus_{\sigma \in Irr(G)} n_\sigma \cdot \sigma$ are two decompositions of π, then $m_\sigma = n_\sigma$ for every σ.

b. A representation π is irreducible if and only if $< \pi, \pi >= 1$.

4. Let V be a finite dimensional vector space over a field $\mathbb{F}$ and let $T : V \to V$ be a linear transformation. Suppose (π, G, V) is a representation of a group G and $V = \bigoplus_{\rho \in Irr(G)} V_\rho$ is a decomposition of V into isotypic components. Suppose that T commutes with π. That is, $\pi(g) \circ T = T \circ \pi(g)$ for every $g \in G$. Show that $T(V_\rho) \subset V_\rho$ for every ρ.

5. Let V be a finite dimensional vector space over $\mathbb{C}$ and let $T : V \to V$ be a linear transformation. Suppose we have a representation (π, G, V) of a group G such that T commutes with π. Further, suppose $V = \bigoplus_{\rho \in Irr(G)} m_\rho \cdot V_\rho$, where m_ρ is 0 or 1 for every ρ. Show that T is diagonalizable.

Good Luck!