Math 741 - Fall 2015

Homework 1 - Notion of Group & Symmetry

Presentations: Mon Sep. 14 - 2015 5pm at VV 9th floor

1. **Uniqueness.** Let $(G, *, 1_G)$ be a group.

   a. Uniqueness of identity. Show that if $e \in G$ satisfies $e * g = g * e = g$ for all $g \in G$ then $e = 1_G$.

   b. Uniqueness of inverse. Show that if $g' \in G$ satisfies $g' * g = g * g' = 1_G$ and similarly $g'' \in G$ satisfies $g'' * g = g * g'' = 1_G$, then $g' = g''$.

2. **Subgroups.** Let $(G, *, 1_G)$ be a group. A subset $H \subseteq G$ is said to be a subgroup of G, denoted $H < G$, if there is a distinguished element $1_H \in H$ such that $(H, *, 1_H)$ is a group. Suppose H is a subgroup of G. Show that $1_H = 1_G$.

3. **Examples of Groups.**

   a. Let $<,>$ denote the standard inner product on $\mathbb{R}^3$. Consider the set

$$O(3) = \{A \in GL_3(\mathbb{R}) | < Au, Av >=< u, v > \text{ for all } u, v \in \mathbb{R}^3\}.$$

   We know that $(O(3), *, I_3)$ forms a group, where $*$ denotes matrix multiplication. Define SO(3) to be the set

$$SO(3) = \{B \in O(3) | \det(B) = 1\}.$$

   Show that SO(3) is a subgroup of O(3).

   b. Let $X_n = \{1, \ldots, n\}$, and define S_n to be the collection of all bijections $\sigma : X_n \to X_n$. Show that $(S_n, \circ, id)$ is a group, where $\circ$ denotes function composition and id is the function defined by $id(x) = x$ for all $x \in X_n$. Find $\#S_n$, where $\#$ denotes cardinality.

   c. An element $\tau \in S_n$ is called a transposition if it interchanges two elements of X_n and leaves the rest fixed. It is a fact that any σ in S_n can be decomposed into a product of transpositions. Such a decomposition is not unique, but if $\sigma = \tau_1 \circ \ldots \circ \tau_m$ and $\sigma = \tau'_1 \circ \ldots \circ \tau'_k$ are two decompositions, then $m - k$ is even. Define A_n to be the subset of S_n consisting of elements which can be decomposed into an even number of transpositions. Show that A_n is a subgroup of S_n, and find $\#A_n$.

4. **Homomorphims.** Let $(G, *_G, 1_G)$ and $(H, *_H, 1_H)$ be groups. A function $\phi : G \to H$ is said to be a homomorphism if $\phi(g_1 *_G g_2) = \phi(g_1) *_H \phi(g_2)$ for all $g_1, g_2 \in G$. Let $\phi : G \to H$ be a homomorphism.

   a. Let $ker(\phi) = \{g \in G | \phi(g) = 1_H\}$. Show that $ker(\phi)$ is a subgroup of G.

   b. A subgroup N of a group G is said to be normal, denoted $N \triangleleft G$, if it is the kernel of some homomorphism. Show that N is normal if and only if $g * N = N * g$ for all $g \in G$.

   c. Show that ϕ is one-to-one if and only if $ker(\phi) = \{1_G\}$.

5. **Group Actions.** Let $(G, *, 1_G)$ be a group and X a set. An action of G on X is a map $\bullet : G \times X \to X$ sending $(g, x) \mapsto g \bullet x$ such that $(g * h) \bullet x = g \bullet (h \bullet x)$ and $1_G \bullet x = x$ for all $g, h \in G$ and $x \in X$.

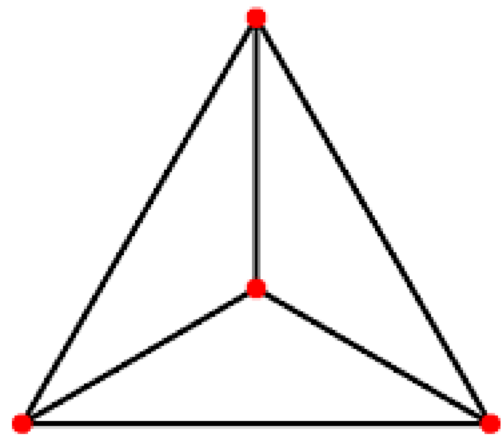
   a. Let G be a group and $\bullet$ an action of G on a set X. We say that a subset $Y \subset X$ is G-invariant if $g \bullet y \in Y$ for all $g \in G$ and $y \in Y$. Denote by $Aut(Y)$ the set

$$Aut(Y) = \{\sigma : Y \to Y | \ \sigma \text{ is a bijection}\}.$$

   Show that $(Aut(Y), \circ, id)$ is a group, where $\circ$ denotes function composition and id is the function defined by $id(y) = y$ for all $y \in Y$.

   b. With G and $Y \subset X$ as above, consider the map r sending $g \in G$ to the function $r[g]$ on Y defined by $r[g](y) = g \bullet y$, for every $y \in Y$. Show that $r[g] \in Aut(Y)$ and that $r : G \to Aut(Y)$ is a homomorphism.

6. **Symmetry Group of the Tetrahedron.** Define the Tetrahedron $\mathcal{T} \subset \mathbb{R}^3$ to be the figure below (centered at the origin).



a. Let $T = \{A \in SO(3) | A(\mathcal{T}) = \mathcal{T}\}$. Show that T is a subgroup of $SO(3)$.

b. Show that $\#T = 12$.

c. Define $Y \subset \mathcal{T}$ to be the set of vertices. Observe that T acts on $\mathcal{T}$ according to $A \bullet v = Av$, where Av denotes the standard action of a matrix A on a vector v. Explain why Y is T-invariant in the sense of exercise 5.

d. Note that part c, combined with exercise 5 part b, imply that we have a homomorphism $r : T \to Aut(Y)$ sending $A \in T$ to the map $r[A]$ on Y, defined by $r[A](y) = Ay$ for every $y \in Y$. Show that r is one-to-one.

e. Deduce that $T \cong A_4$ (you may use the fact that S_n has a unique subgroup H such that $\#S_n/\#H = 2$).

Good Luck!