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Notes Project
Math 541

Lecture 1: March 9th, 2011

Definition: A group C is called Cyclic if $\exists c \epsilon C$ such that $C = \{c^k; k = z, k\epsilon\mathbb{Z}\}$, where $c^k = c \cdot c \cdot \dots \cdot c$, multiplied k times if k>o, then c is also called a generator for C.

Theorem: Suppose $\Gamma \subset O(2), \#\Gamma < \infty \ (finite)$.

$$Then\ \Gamma \cong C_n\ for\ some\ n \geq 1, or\ \Gamma \text{ is isomorphic to } \mathrm{D_n}\ for\ some\ n \geq 1$$

Category of Groups:

(1) Objects=Groups
(2) Relationships between objects: These relationships are special functions (morphisms) and between groups, and are called homomorphisms. Then, we have a collection of groups and the homomorphisms between them.

Definition: A function $\varphi: G \to H$ between group G and group H is called a homomorphism if it sends a product to a product:

$$\varphi(g_1 \cdot g_2) = \varphi(g_1) \cdot \varphi(g_2) \qquad \forall g_1, g_2 \epsilon G.$$

Examples:

1. det: $GL_2(\mathbb{R}) \to \mathbb{R}^*$

   A$\mapsto$ $\det(A)$

By the determinant theorem det(A$\cdot B) = \det(A) \cdot \det(B)$, we know it is a homomorphism.

2. $det: GL(n, \mathbb{R}) \to \mathbb{R}^*\ is\ a\ homomorphism.$

3. $\mathrm{d}et: O(2, \mathbb{R}) \to \{\pm 1\} is\ a\ homomorphism$

4. Suppose G is a group, acting on a set X, such that $Y \subset C, an\ invariant\ subset$; i.e. all elements g$\epsilon G\ takes\ Y\ onto\ itself, namely\ \forall g\epsilon G, g(Y) = Y$. Then we have a bijection r[g]: Y→ Y, r[g](y)=g$\cdot\ y, \forall y \in Y, \forall g\epsilon G$

Claim: The map r: G→ Aut(Y), given by g↦r[g], is a homomorphism.

5. Sign Homomorphism: sgn: $S_n \to \{\pm 1\}\ \ where\ S_n = Aut\{(1, \dots, n)\}$

   Discriminant: n=2 $\qquad D(X_1, X_2) = X_2 - X_1$
   Suppose $\sigma \in S_2$, then ${}^{\sigma}D(X_1, X_2) = \left(DX_{\sigma^{-1}(1)}, X_{\sigma^{-1}(2)}\right) = X_{\sigma^{-1}(2)} - X_{\sigma^{-1}(1)}$

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If $\sigma \in S_n$, then Note that: $^{\sigma}D = \begin{cases} -D \quad , if\ \sigma = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} : 1,2, \ldots \ldots, n \\ D \quad , if\ \sigma = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} : \sigma(1), \sigma(2), \ldots \sigma(n) \end{cases}$

Then $^{\sigma}D = sgn(\sigma) \cdot D = \pm 1 \cdot D.$

Lecture 2: March 21st, 2011

Definition: A function $\varphi: G \to H$ between group G and group H, is called a <u>homomorphism</u> if it sends a product to a product: $\varphi(g_1 \cdot g_2) = \varphi(g_1) \cdot \varphi(g_2) \quad \forall g_1, g_2 \epsilon G.$

Example: Sign homomorphism:

If the discriminant is n=2, so $D(X_1, X_2) = X_2 - X_1$ then we can suppose $\sigma \in S_2 = Aut\{1,2\}$.

Then we can apply σ on D; i.e. $^{\sigma}D(X_1, X_2) = \left(DX_{\sigma^{-1}(1)}, X_{\sigma^{-1}(2)}\right) = X_{\sigma^{-1}(2)} - X_{\sigma^{-1}(1)}$

So if $\sigma = (\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix})$ sthen $^{\sigma}D = x_1 - x_2$= -D.

$^{\sigma}D = sgn(\sigma) \cdot D$

sgn(σ) $= \begin{cases} 1 & \sigma = id \\ -1 & \sigma \neq id \end{cases}$ }

Claim: sgn: $S_2 \to \{\pm 1\}$ *is a homomorphism*

Proof: Now, we wish to show the most interesting case is when

Sgn($\sigma \cdot \sigma) = sgn(\sigma) \cdot sgn(\sigma) \quad for\ \sigma = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

Multiplication Table:

	identity	$\sigma = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$
Identity	Identity	σ
σ	σ	Identity

Show: sgn($\sigma * \sigma) = -1 * -1 = 1$

$\sigma * \sigma = id$

$sgn(\sigma) = 1$

So, since it holds, we know it is a homomorphism.

Now: We wish to how it is true with three variables; i.e. when the discriminate is equal to three.

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N=3, $D = (x_1, x_2, x_3) = (x_3 - x_2) * (x_2 - x_1) * (x_3 - x_1)$

Where $\sigma \in S_3 = Aut\{(1,2,3)\},\ and\ this\ suggests\ that\ it\ is\ 1 \div 1\ \&\ onto.$

Define a new polynomial:

$$^{\sigma}D(x_1, x_2, x_3) = D\left(x_{\sigma(1),}\, x_{\sigma(2),}\right) = \left(x_{\sigma^{-1}(3)} - x_{\sigma^{-1}(2)}\right) * \left(x_{\sigma^{-1}(2)} - x_{\sigma^{-1}(1)}\right) * \left(x_{\sigma^{-1}(3)} - x_{\sigma^{-1}(1)}\right).$$

Claim: (a) $^{\sigma}D = sgn(D)\ \cdot D \quad where\ sgn(\sigma) = \pm 1.$

(b) sgn: $S_3 \to \{\pm 1\}\ \ is\ a\ homomorphism\ (and\ is\ 1 \div 1\ \&\ onto).$

Proof: (a)
$If\ \sigma\ is\ a\ transposition,$
$then\ sgn(\sigma)\ is\ equal\ to -1, then\ we\ will\ \ argue\ that\ there\ is\ a\ formula\ where$

$$sgn(\sigma) = (-1)^k$$

where k is the number of all pairs such that (i,j), $1 \leq i \leq j \leq n,\ \ but\ \sigma(j) < \sigma(i).$

Therefore we know that $^{\sigma}D = sgn(D)\ \cdot D \quad where\ sgn(\sigma) = \pm 1.$

Proof (b) $We\ wish\ to\ show\ sgn\colon S_3 \to \{\pm 1\}\ \ is\ a\ homomorphism$ where $\sigma, \tau \in S_3$,

then compute $sgn(\sigma * \tau) \overset{?}{=} sgn(\sigma) * sgn(\tau).$

(note: * is the composition of functions)

$$\sigma * \tau * D = {}^{\sigma * \tau}D = {}^{G}(\,{}^{\tau}D) = {}^{\sigma}(sgn(\tau) * D) = sgn(\tau) * {}^{\sigma}(D) = sgn(\tau) * sgn(\sigma) * D$$

So, by this we obtained $sgn(\sigma * \tau) = sgn(\tau) * sgn(\sigma) = sgn(\tau) * sgn(\sigma).$

Because $sgn(\sigma) = \pm 1\ and\ sgn(\tau) = \pm 1,$ we know that $sgn(\sigma * \tau) = \ sgn(\sigma) * sgn(\tau)$, so we proved our claim. Therefore we know sgn: $S_3 \to \{\pm 1\}\ \ is\ a\ homomorphism.$