Math 541 - Notes from February 11th & 14th (covering HW2)

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1 Rotational Symmetries

a. Consider the collection R of all matrices of the form:

$$r_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \theta \in \mathbb{R}$$

Show that this set, endowed with matrix multiplication ($\circ$), forms a group.

Main Point: The important property to prove is closure under the group operation, which we can establish by familiar trigonmetric identities.

(i) Closure under the group operation.

This can be shown by considering the following statement:

$$r_{\theta 1} \circ r_{\theta 2} \stackrel{?}{=} r_{\theta 1 + \theta 2}$$

Showing this statement to be true is equivalent to showing that the set R is closed under matrix multiplication, by the following reasoning.

Proof:

Since $\circ$ is matrix multiplication, this is equivalent to the statement:

$$r_{\theta 1} \circ r_{\theta 2} = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} \times \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix}$$

This expands to:

$$r_{\theta 1} \circ r_{\theta 2} = \begin{pmatrix} (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) & (-\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2) \\ (\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) & (-\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2) \end{pmatrix}$$

By the angle-sum trigonometric identities, the components of the matrix are equal to:

$$r_{\theta 1} \circ r_{\theta 2} = \begin{pmatrix} \cos(\theta_1 + \theta_2) & -\sin(\theta_1 + \theta_2) \\ \sin(\theta_1 + \theta_2) & \cos(\theta_1 + \theta_2) \end{pmatrix}$$

Which proves the original statement:

$$r_{\theta 1} \circ r_{\theta 2} = r_{\theta 1 + \theta 2}$$

Additionally, we must verify that the identity matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, which is the identity of matrix multiplication, is within R. Since $\cos(0) = 1$ and $\sin(0) = 0$, we can express the identity matrix as $\begin{pmatrix} \cos(0) & -\sin(0) \\ \sin(0) & \cos(0) \end{pmatrix}$, which is in R.

(ii) Associativity of the group operation.
This follows from the known properties of matrix multiplication.

(iii) Existence of an identity element 1_R such that $\forall r \in R, \quad 1_R \circ r = r$.
A separate proof of this is not necessary as $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is known to be the identity of matrix multiplication and is also known to be in R per the proof in part (i).

(iv) Existence of an inverse in R for every element in R.
Since we have already proved the identity $r_{\theta 1} + r_{\theta 2} = r_{\theta 1 + \theta 2}$, we can deduce that the inverse of r_θ is $r_{-\theta}$. Since $r_{-\theta}$ is also in R (which can be trivially shown by noting that cos is an even function while sin is an odd function, and therefore both have straightforward rules for propagating the negation of their argument), every $r \in R$ must have an inverse also in R.

b. Define what it means for a group to act on a set.

Definition: Let G be a group and X be a set. A map $\bullet : G \times X \to X = (g, x) \to (g \bullet x)$ is called an action of G on X if:

(i) $\forall g, h \in G, \, x \in X \quad (g \circ h) \bullet x = g \bullet (h \bullet x)$.

(ii) $\forall x \in X \quad 1_G \bullet x = x$.

c. Take G to be the group of rotational symmetries R and X to be $\mathbb{R}^2$. Consider $\bullet : R \times \mathbb{R}^2 \to \mathbb{R}^2$, the map which sends a pair (r_θ, v) to the vector $r_\theta(v)$ (i.e. the rotation of v by angle θ). Show that this map satisifes the axioms for the action of a group on a set.

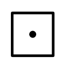
Main Point: The important properties follow from known properties of matrix multiplication.

(i) Associativity between the the group operation and $\bullet$.
We must show that $\forall r_{\theta 1}, r_{\theta 2} \in R, v \in \mathbb{R}^2$, $(r_{\theta 1} \circ r_{\theta 2}) \bullet v = r_{\theta 1} \bullet (r_{\theta 2} \bullet v)$. Fortunately, this follows directly from the associativity of matrix multiplication, since all of $r_{\theta 1}$, $r_{\theta 2}$, and v are matrices (a 2 by 1 matrix in the case of v).

(ii) Group identity works as action identity.
We must show that $\forall v \in \mathbb{R}^2 \quad 1_R \bullet v = v$. Again, this follows directly from the known properties of matrix multiplication.

2 Rotational Symmetries of a Square (□)

a. Write down the diagram of the standard square, with vertices at $(\pm 1, \pm 1)$.



b. Compute the set C_4 of all rotational symmetries of the square.

Main Point: We may calculate the members of C_4 directly.

$C_4 = \{r_\theta \in R \mid r(\Box) = \Box\} = Stab_G(\Box)$

$\{r_0, r_{90}, r_{180}, r_{270}\}$ are all the rotational symmetries of a square and form a group.

c. Show that $(C_4, \circ, I)$ (where $\circ$ is matrix multiplication and I is the identity rotation) forms a group.

Main Point: The most important property to prove is closure under the group operation. The other properties either follow from the known properties of matrix multiplication, or can be shown by direct calculation.

(i) Closure under matrix multiplication.

This can be proved by exhaustion, since there are only 4 members of C_4 and thus only $4! = 24$ ways to add them (not taking into account the fact that we know thhat the multiplications are commutative). It can also be proved by recalling the identity $r_{\theta 1} + r_{\theta 2} = r_{\theta 1 + \theta 2}$, and noting that we know that the addition of $\{0, 90, 180, 270\}$ is closed under addition $mod\, 360$.

(ii) Existence of an identity element.

We know that $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is an identity for matrix multiplication and corresponds to $r_0$from problem 1, and therefore it is also in C_4.

(iii) Associativity of the group operation.

Since associativity is a known property of matrix multiplication, we do not need to prove this for C_4 in particular.

(iv) Existence of inverse elements.

By exhaustion we can determine that:

$(r_0)^{-1} = r_0 \qquad (r_{90})^{-1} = r_{270} \qquad (r_{180})^{-1} = r_{180} \qquad (r_{270})^{-1} = r_{90}$

Therefore every element in C_4 has an inverse also in C_4.

3 G-symmetries

Problem: Let $(G, \circ, 1)$ be a group acting on a set X . For a subset $Y \subset X$ we define the set of G-symmetries of Y to be the subset $Stab_G(Y) \subset G$ given by $Stab_G(Y) = \{g \in G;\ g(Y) = Y\}$ where $g(Y) = \{g \bullet y;\ y \in Y\}$. The set $Stab_G(Y)$ is called the stabilizer of Y in G.

a. Show that $Stab_G(Y)$ is a subgroup of G.

Main Point: Verification following from the defintions of stabilizer, subgroup, and equality of sets.

(i) Closure under the group operation.

$\forall g, h \in Stab_G(Y)$, $(g \circ h)(Y) = g(h(Y)) = g(Y) = Y$, by the definition of a stabilizer. Therefore $(g \circ h)$ is also in the stabilizer of Y, and the subgroup is closed under the group operation.

(ii) Associativity of the group operation.

Since G is already known to be a group, we know that the group operation must be associative.

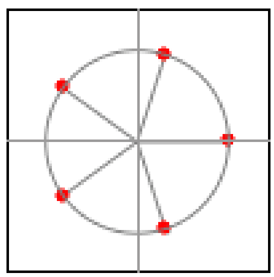
(iii) Identity of the group operation.

1_G acts as the identity of the group action, as previously established. Therefore, $1_G(Y) = Y$. Therefore, $Stab_G(Y)$ must include 1_G.

(iv) Closure under inversion.

It follows from the definition of stabilizer that $\forall g \in Stab_G(Y)$, $\forall y \in Y$, $\exists y' \in Y$ $s.t.$ $g \bullet y' = y$. If we multiply by g^{-1}, we show that $\forall y \in Y$, $\exists y' \in Y$ $s.t.$ $g^{-1} \bullet y = y'$. Therefore $g^{-1}(Y) \subset Y$. To show that $Y \subset g^{-1}(Y)$, we note that $\forall y \in Y$, $\exists y' \in Y$ $s.t.$ $y' = g \bullet y$, by the definition of stabilizer. If we multiply both sides by g^{-1}, we show that $\forall y \in Y$, $\exists y' \in Y$ $s.t.$ $g^{-1} \bullet y' = y$. Therefore, $Y \subset g^{-1}(Y)$.

b. Consider the standard five-sided polygon Y in the plane $\mathbb{R}^2$ with vertices at the points satisfying the equation $z^5 = 1$, where points are considered to be complex numbers $z \in \mathbb{C}$. Draw its diagram.



c. Compute the group $C_5 = Stab_R(Y) = \{r_\theta \in R;\ r_\theta(Y) = Y\}$ of rotational symmetries of Y.

Main Point: We can enumerate C_5 by direct calculation.
The group of rotational symmetries is isomorphic to the set of fifth roots of unity endowed with complex multiplication, with 1 as the identity. Alternatively, it may be viewed as the set of the rotation matrices $\{r_0, r_{72}, r_{144}, r_{216}, r_{288}\} \subset R$, endowed with matrix multiplication and r_0 as identity.