Math 541 Spring 2011
Preparation for the Mid-Term of 11/03/11

Remark. Answers should be written in the following format:
i) Statement and/or Result.
ii) Main points that will appear in your explanation or proof or computation.
iii) The actual explanation or proof or computation.

1. (20) *Definitions.*
   (a) Define the notion of a group $(G, *, 1_G)$.
   (b) Define the notion of a group G acting on a set X. Denote this action by $\cdot$.

2. (40) *Theory.* Suppose we have a group G acting on a set X.
   (a) Suppose $Y \subset X$ subset. For an element $g \in G$ we define the set $g(Y) = \{g \cdot y;\ y \in Y\}$. Show that the subset $H \subset G$ given by
   $$H = Stab_G(Y) = \{g \in G;\ g(Y) = Y\},$$
   is a subgroup of G.
   (b) Suppose that $Y \subset X$ is invaraint under the action of G, i.e., for EVERY $g \in G$ we have $g(Y) = Y$.
      1. Show that, in this case, for every $g \in G$, the map
      $$\begin{aligned} r[g] &: Y \to Y, \\ r[g](y) &= g \cdot y, \end{aligned}$$
      is a bijection. In particular, $r[g] \in Aut(Y)$.
      2. Show that the map
      $$\begin{aligned} r &: G \to Aut(Y), \\ g &\mapsto r[g], \end{aligned}$$
      is homomorphism, i.e., satisfies $r[g_1 * g_2] = r[g_1] \circ r[g_2]$, where $\circ$ denotes composition of functions.

3. (40) *Application.* Denote by T the natural equilateral triangle around the origin in $\mathbb{R}^2$.
   (a) Define the natural homomorphism
   $$r : D_3 = Stab_O(T) \to S_3 = Aut(\{a, b, c\}),$$
   where O denotes the group of orthogonal transformations of $\mathbb{R}^2$.
   (b) Show that this r is an isomorphism, i.e., it is one-to-one and onto.

Good Luck!