Friday, January 28, 2011

Definition: *A* **Group** *is a triple denoted by* $(G, \cdot, 1_G)$, or $(G, \cdot, 1)$, *where* G *is a set,* $\cdot$ *is a binary operation defined by* $\cdot : G \times G \to G, (g_1, g_2) \mapsto g_1 \cdot g_2,\ \forall\, g_1, g_2 \in G$, *and* $1_G \in G$ *is called the identity, that satisfies the following 3 axioms:*

i) *Associativity:*
$\forall\, g_1, g_2, g_3 \in G$ *such that the following holds:*

$$(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$$

ii) *Identity:*
$\forall\, g \in G, \exists\, 1_G \in G$, *such that the following holds:*

$$1_G \cdot g = g \cdot 1_G = g$$

iii) *Inverses:*
$\forall\, g \in G, \exists\, g' \in G$ *such that the following holds:*

$$g \cdot g' = g' \cdot g = 1_G$$

Homework problem 2) a.

Claim: If $e, e' \in G$ such that $e \cdot g = g \cdot e = g, \forall\, g \in G$ and $e' \cdot g = g \cdot e' = g, \forall\, g \in G$, then $e = e'$.

Proof:

$e \cdot g = g$ (property of e)

$e' \cdot g = g$ (property of e')

Combining these equations gives

$e \cdot g = e' \cdot g$

Since $g \in G, \exists\, g'$ such that $g' \cdot g = g \cdot g' = 1_G$.

Multiplying, $e \cdot g = e' \cdot g$, by g' on the right.

$(e \cdot g) \cdot g' = (e' \cdot g) \cdot g'$ (Associativity)

$e \cdot (g \cdot g') = e' \cdot (g \cdot g')$ (Inverse property)

$e \cdot 1_G = e' \cdot 1_G$ (Identity Property)

$e = e'$ ∎

We used the properties given (e and e') and the axiomatic properties of a group.

Alternatively, we could have done the following using properties of the identity.

$$e \cdot e' = e' \cdot e = e \quad \text{and}$$

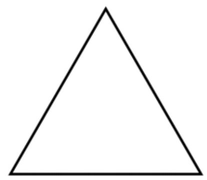
$$e \cdot e' = e' \cdot e = e'$$

By combining these two equations we obtain:

$$e' = e \cdot e' = e \text{ or}$$

$$e = e' \blacksquare$$

Now, we discuss the triangle again.



Consider the group $(C_3, \circ, I)$, the group of rotational symmetries about the origin of an equilateral triangle. Where the binary operation, $\circ$, is matrix multiplication of the rotational matrices that make up C_3.

$$C_3 = \left\{ r \in R \,|\, r\left(\triangle \right) = \triangle \right\} = \left\{ r_0, r_{\frac{2\pi}{3}}, r_{\frac{4\pi}{3}} \right\}$$

<u>Claim</u>: $(C_{3,} \circ, I)$ is a group.

<u>Proof</u>:

As a preliminary step, we check closure that the product of two rotations is, again, a rotation. This can be done using matrix multiplication as shown in the last lecture:

Let $r_{\theta_1}, r_{\theta_2} \in C_3$

$r_{\theta_1} \circ r_{\theta_2} = r_{\theta_1+\theta_2} \in C_3$ via matrix multiplication

$$r_{\theta_1} \circ r_{\theta_2} = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} \circ \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix} =$$
$$\begin{pmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & -\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2 \\ \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 & -\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2 \end{pmatrix} = r_{\theta_1+\theta_2} \in C_3 \quad \blacksquare$$

i) Associativity:

$\forall\, r_{\theta_1}, r_{\theta_2}, r_{\theta_3} \in C_3$ need to show that

$r_{\theta_1} \circ (r_{\theta_2} \circ r_{\theta_3}) = (r_{\theta_1} \circ r_{\theta_2}) \circ r_{\theta_3}$

Since $C_3 \subset R$, this follows via matrix multiplication as shown in the last lecture.

ii) Identity:

This is the familiar 2x2 identity matrix from linear algebra.

iii) Inverses:

Need to show every element of C_3 has an inverse.

i.e. $\forall\, r_\theta \in C_3\ \exists\, r_\theta' \in C_3$ such that:

$$r_\theta \circ r_\theta' = r_\theta' \circ r_\theta = I$$

This claim can be proven by checking each element of C_3.

$$r_0 \circ r_0 = r_0$$
$$r_{\frac{2\pi}{3}} \circ r_{\frac{4\pi}{3}} = r_{\frac{4\pi}{3}} \circ r_{\frac{2\pi}{3}} = r_0$$
$$r_{\frac{4\pi}{3}} \circ r_{\frac{2\pi}{3}} = r_{\frac{2\pi}{3}} \circ r_{\frac{4\pi}{3}} = r_0$$

These 3 axioms prove C_3 is a group. ■