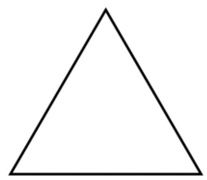
Wednesday, January 26, 2011

2 Goals of Today's Lecture:

1) Introduce language of groups.
2) Explain symmetries in the plane, $\mathbb{R}^2$, and space, $\mathbb{R}^3$.

Example:

Consider an equilateral triangle in the plane, $\mathbb{R}^2$.



Question: What are the symmetries of the triangle?

Before we can discuss the relevant symmetries of the triangle, we need to remind ourselves of the definition of a **linear operator** in order to characterize the symmetries unambiguously.

Definition: *A* **linear operator** *on a vector space, V, is a map, $A: V \to V$, such that the following axioms hold.*

1) *$\forall\, u, v \in V$ the following holds:*

$$A(u+v) = A(u) + A(v)$$

2) *$\forall\, u \in V$ and $c \in \mathbb{R}$ the following holds:*

$$A(cu) = cA(u)$$

Still, keeping in mind the triangle above, define a vector space $V = \mathbb{R}^2$, and define a linear operator, A, of the following form:

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, such that, $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, where $\begin{pmatrix} x \\ y \end{pmatrix} \in V$.

Now, consider the rotational symmetries about the origin of the equilateral triangle above. We define the following **linear operator** on $\mathbb{R}^2$:

$\forall\, \theta \in \mathbb{R}$, where θ is the angle of rotation about the origin, denoted by a dot in the center of the triangle, consider the following **linear operator**, $r_\theta: \mathbb{R}^2 \to \mathbb{R}^2$, defined as follows:

$$r_\theta \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}, \forall \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$$

Consider the set $R \coloneqq \{r_\theta,\ \theta \in \mathbb{R}\}$, where R is the set of rotations of the equilateral triangle in the plane $\mathbb{R}^2$ about the origin by an angle θ.

Now, consider the natural operation, $\circ$, such that:

$$\circ : R \times R \to R$$
$$(r_{\theta_1}, r_{\theta_2}) \longmapsto r_{\theta_1} \circ r_{\theta_2}$$

Where $\circ$ denotes the standard matrix multiplication.

<u>Claim</u>: If $r_{\theta_1}, r_{\theta_2} \in R$ then $r_{\theta_1} \circ r_{\theta_2} \in R$.

<u>Proof</u>: Indeed, calculate the following:

$r_{\theta_1} \circ r_{\theta_2} = r_{\theta_1+\theta_2} \in R$ via matrix multiplication

$$r_{\theta_1} \circ r_{\theta_2} = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} \circ \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix} =$$
$$\begin{pmatrix} \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 & -\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2 \\ \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 & -\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2 \end{pmatrix} = r_{\theta_1+\theta_2} \in R$$

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In addition, there exists a special element, $1_R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in R$.

And the triple $(R, \circ, 1_R)$ satisfies the following properties:

1) *Associativity:*
   $\forall\, r_{\theta_1}, r_{\theta_2}, r_{\theta_3} \in R$ *such that the following holds:*
   $$r_{\theta_1} \circ (r_{\theta_2} \circ r_{\theta_3}) = (r_{\theta_1} \circ r_{\theta_2}) \circ r_{\theta_3}$$
   This holds via matrix multiplication as done above.

2) *Identity:*
   $\forall\, r_\theta \in R, \exists\, 1_R \in R$ *such that the following holds:*
   $$1_R \circ r_\theta = r_\theta \circ 1_R = r_\theta$$
   This holds via matrix multiplication as done above using the 2x2 identity matrix obtained from $\theta = 0\ degrees$.

3) *Inverses:*
   $\forall\, r_\theta \in R, \exists\, r_{-\theta} \in R$ *such that the following holds:*
   $$r_\theta \circ r_{-\theta} = r_{-\theta} \circ r_\theta = 1_R$$
   This holds via matrix multiplication as well.

We have just proven that $(R, \circ, 1_R)$ forms a **group**, which satisfies those axioms listed above and, in general, the following axioms below.

Definition: *A* **Group** *is a triple denoted by* $(G, \cdot, 1_G)$, or $(G, \cdot, 1)$, *where* G *is a set,* $\cdot$ *is a binary operation defined by* $\cdot : G \times G \to G, (g_1, g_2) \mapsto g_1 \cdot g_2,\ \forall\, g_1, g_2 \in G$, *and* $1_G \in G$ *is called the identity, that satisfies the following 3 axioms:*

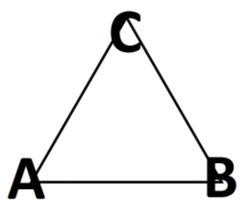
i) *Associativity:*
$\forall\, g_1, g_2, g_3 \in G$ *such that the following holds:*
$$(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$$

ii) *Identity:*
$\forall\, g \in G, \exists\, 1_G \in G$, *such that the following holds:*
$$1_G \cdot g = g \cdot 1_G = g$$

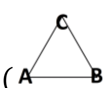
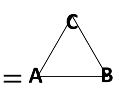
iii) *Inverses:*
$\forall\, g \in G, \exists\, g' \in G$ *such that the following holds:*
$$g \cdot g' = g' \cdot g = 1_G$$

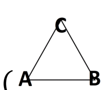
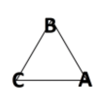
Our above example is the rotational group, $(R, \circ, 1_R)$.

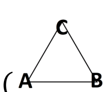
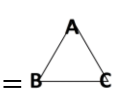
Let's now revisit the triangle:



And label the vertices as above, and apply non-trivial counter-clockwise rotations of $\frac{2\pi}{3}$ radians about its center.
We obtain the following:

$r_0($  $) =$ 

$r_{\frac{2\pi}{3}}($  $) =$ 

$r_{\frac{4\pi}{3}}($  $) =$ 

Now, consider $Y \subset \mathbb{R}^2$. Where Y is the triangle with vertices A, B, and C.

i) Consider the collection of all $r_\theta \in R$ that takes $Y \longmapsto Y$, (preserves the set Y).
i.e.

$$r_\theta(Y) \coloneqq \{r_\theta(y) \mid y \in Y\}$$

We say, "r_θ preserves Y." If $r_\theta(Y) = Y, r_\theta \in R$.
These rotations, r_θ, are the rotational symmetries of Y.

ii) We will say Y has rotational symmetries if there exists $r_\theta \neq r_0$ (the trivial rotation) that preserves Y.

An example of this is the following set (rotational group) C_3:

$$C_3 = \left\{r_0\,, r_{\frac{2\pi}{3}}\,, r_{\frac{4\pi}{3}}\right\}$$

Claim: C_3 is a **group**. Denoted by $(C_3, \circ, I)$ where I is the 2x2 identity matrix and $\circ$ is the binary operation of matrix multiplication.

Sketch of the Proof: This amounts to taking compositions of rotations of the triple as shown above via matrix multiplication in the plane $\mathbb{R}^2$. The identity is the 2x2 identity matrix. The inverses are calculated by noticing that the inverse of rotation by $\frac{2\pi}{3}$ is the inverse of rotation by $\frac{4\pi}{3}$ and vice versa. ■