Math 541 Spring 2011
Homework#8, 7/04/11— Orbits,Cosets, Lagrange's Theorem, Fermat's Little Theorem

Remark. Answers should be written in the following format:
0) Statement and/or Result.
i) Main points that will appear in your explanation or proof or computation.
ii) The actual explanation or proof or computation.

1. *Orbits.* Describe the set of orbits (=equivalence classes) $G\backslash X$ in the following cases:
   (a) The set $X = \mathbb{R}^2$, the group $G = SO(2)$ the rotations of the plane, and the action is the natural action, via action of a matrix on a vector.
   (b) The set $X = \mathbb{R}^2 - \{0\}$, the group $G = \mathbb{R}^* = \mathbb{R} - \{0\}$, and action is the scaling action of G on X via $a \cdot (x, y) = (a \cdot x, a \cdot y)$.
   (c) The set X is the collection of equilateral triangles around the origin in the plane $\mathbb{R}^2$, the group $G = SO(2)$, and the action is the natural one, induced from the action of rotations on the plane.

2. *Cosets.* In each of the following compute the set of left cosets G/H, and if G is finite verify Lagrange's Theorem $\#G = [G : H] \cdot \#H$,where $[G : H] = \#(G/H)$, called the index of H in G.
   (a) $G = S_3$, $H = A_3$ the group of even permutations of three letters.
   (b) $G = S_n$, $H = A_n$.
   (c) $G = D_3$, $H = C_3$, Where D_3 is the dihedral group of order 3 and C_3 is its subgroup (cyclic of order 3) of rotations.
   (d) $G = \mathbb{R}$, $H = \mathbb{Z}$.
   (e) $G = O(2)$, $H = SO(2)$.

3. Fermat's little theorem. Let p be a prime number. Denote by $\mathbb{Z}_p^* = \{1, ..., p - 1\}$.
   (a) Show that $\mathbb{Z}_p^*$ is a groups with operation of multiplication modulo p.
   (b) Show that for every element $x \in \mathbb{Z}_p^*$ we have $x^{p-1} = 1$ modulo p. Hint: Use Lagrange'sTheorem with the groups $G = \mathbb{Z}_p^*$, and $H =< x >$ the subgroup generated by x.
   (c) Conclude with Fermat's little Theorem: For $x \in \mathbb{Z}_p$, we have $x^p = x \bmod(p)$.

Remarks
- You are very much encouraged to work with other students. However, submit your work alone.
- I will be happy to help you with the home works. Please visit me in my office hours.

Good Luck!