Math 541 Spring 2011
Homework#7, 5/04/11— Normal Subgroups, Lagrange's Theorem, Groups of Prime Order

Remark. Answers should be written in the following format:
0) Statement and/or Result.
i) Main points that will appear in your explanation or proof or computation.
ii) The actual explanation or proof or computation.

1. *Normal subgroups.* Recall that a subgroup N of a group G is called <u>normal</u> if $g \cdot N \cdot g^{-1} = N$ for every $g \in G$. In this case we denote $N \lhd G$.
   (a) Show that if $\varphi : G \to H$ is an homomorphism of groups, then $N = \ker(\varphi) \lhd G$.
   (b) Show that the group R of rotations of the plane is a normal subgroup of the group O of orthogonal transformation of $\mathbb{R}^2$.
   (c) Show that the group A_n of even permutations of $\{1, 2, ..., n\}$ is a normal subgroup S_n.
   (d) Find all normal subgroups of S_3.

2. Lagrange's Theorem. Recall that if H is a subgroup of a group G, then $\#H$ divides $\#G$.
   (a) What is the possible cardinality of subgroups of S_4.
   (b) Can you find for each of the possibilities in (a), a subgroup $H \subset G$ with that cardinality?

3. *Classification of all subgroups of prime order.* Let G be a groups of order (cardinality) $\#G = p$, where p is a prime number. Show that G is cyclic of order p. Use the following steps
   (a) Choose a non-trivial element $1 \neq g \in G$, and consider the group $< g >= \{g^k;\ k \in \mathbb{Z}\}$. Show that $< g >= \{1, g, .., g^{d-1}\}$ for some integer $d > 0$. This d is called the order of g, denoted by $d = ord(g)$.
   (b) Since by Lagrange's theorem we have $\# < g > |\#G$, and $\#G = p$, deduce that $< g >= G$.

- You are very much encouraged to work with other students. However, submit your work alone.
- I will be happy to help you with the home works. Please visit me in my office hours.

Good Luck!