Math 541 Spring 2011
Homework#6, 3/04/11— Homomorphisms, Sign, Chinese reminder theorem

Remark. Answers should be written in the following format:
0) Statement and/or Result.
i) Main points that will appear in your explanation or proof or computation.
ii) The actual explanation or proof or computation.

1. *Homomorphism.*
   (a) Let $\alpha : G \to H$, $\beta : H \to K$ be two homomorphisms of groups. Show that the composition $\gamma = \beta \circ \alpha : G \to K$ is also homorphism.
   (b) Show that if $\varphi : G \to H$ is an homomorphism which is a bijection, i.e., isomorphism, then $\varphi^{-1} : H \to G$ is also an homomorphism.
   (c) Suppose G is a group acting on a set X. Consider the map $\alpha : G \to Aut(X)$, given by $g \mapsto \alpha[g]$, where $\alpha[g](x) = g \cdot x$. Show that α is an homomorphism.

2. *The sign homomorphism.* Consider the sign homomorphism $sgn : S_n = Aut(\{1, ..., n\}) \to \{\pm 1\}$.
   (a) Show that every permutation $\sigma \in S_n = Aut(\{1, ..., n\})$ can be written as a product of disjoint cycles
   $$\sigma = c_k \circ ... \circ c_1,$$
   where by a cycle $c \in S_n$ we mean a permutation of the form
   $$c = (a_1\ a_2\ ...\ a_l), \quad 1 \leq a_i \leq n,$$
   i.e., $c(a_1) = a_2, ..., c(a_{l-1}) = a_l$, $c(a_l) = a_1$.
   (b) Decompose the following permutation
   $$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 4 & 5 & 1 & 3 & 8 & 6 & 7 \end{pmatrix} \in S_8,$$
   to a product of disjoint cycles.
   (c) Show that every cycle $c = (a_1\ a_2\ ...\ a_l) \in S_n$ can be written as a product of transpositions.
   (d) Decompose the cycles that appear in 2(b), above, to a product of transpositions.
   (e) Compute the sign $sgn(\sigma)$ for σ above.
   (f) Can you suggest other ways how to compute $sgn(\sigma)$, for $\sigma \in S_n$?

3. *Chinese reminder theorem.* Let m, n be two positive natural numbers which are co-prime, i.e., the greatest common devisor of m and n is 1 (notation: $\gcd(m,n) = 1$). Consider the groups $\mathbb{Z}_{mn} = \{0, 1, ..., mn-1\}$, $\mathbb{Z}_m = \{0, 1, ..., m-1\}$, $\mathbb{Z}_n = \{0, 1, ..., n-1\}$ with the natural addition modulo mn, m, n respectively.

   (a) Show that the map

   $$\begin{array}{rcl} \varphi & : & \mathbb{Z}_{mn} \to \mathbb{Z}_m \times \mathbb{Z}_n, \\ x & \mapsto & (x \mod m, \ x \mod n) \ , \end{array}$$

   is an homomorphism.

   (b) Show that there exist a unique integer $0 \le x \le 98$ such that

   $$\begin{cases} x = 3 \mod 11, \\ x = 7 \mod 9. \end{cases}$$

   Find such an x.

Remarks

- You are very much encouraged to work with other students. However, submit your work alone.
- I will be happy to help you with the home works. Please visit me in my office hours.

Good Luck!