Math 541 Fall 2017
HW6 - Counting, Quotient Groups, Symmetries in Space
Due Friday Dec. 1 2017

Remark. Answers should be written in the following format:
0) Statement and/or Result.
i) Main points that will appear in your explanation or proof or computation.
ii) The actual explanation or proof or computation.

1. *Counting.* Recall that if $\varphi : G \to G'$ is a homomorphism then

$$\#G = \#\ker(\varphi) \cdot \#\operatorname{Im}(\varphi). \tag{1}$$

Consider the finite field with prime p elements $\mathbb{F}_p = \{0, 1, ..., p-1\}$, with $+$ and $\cdot$ modulo p.

   (a) Denote by $GL_n(\mathbb{F}_p) = \{A = (a_{ij}),\ \ n \times n$ matrix with entries $a_{ij} \in \mathbb{F}_p$, and $\det(A) \neq 0\}$. Show that with the operation of matrix multiplication, and identity element $I = I_n$, $GL_n(\mathbb{F}_p)$ is a group. It is called the general linear group of order n over $\mathbb{F}_p$.

   (b) Compute the number of elements in $GL_n(\mathbb{F}_p)$ for $n = 2, 3$.

   (c) Denote by $SL_n(\mathbb{F}_p) = \{A \in GL_n(\mathbb{F}_p);\ \det(A) = 1\}$. Show that with the operation of matrix multiplication, and identity element $I = I_n$, $SL_n(\mathbb{F}_p)$ is a group. It is called the special linear group of order n over $\mathbb{F}_p$.

   (d) Compute the number of elements in $SL_n(\mathbb{F}_p)$ for $n = 2, 3$. Hint: use identity (1) with the homorphism $\det : GL_n(\mathbb{F}_p) \to \mathbb{F}_p^*$.

2. *Quotient groups.* Recall that if G is a group and $N \lhd G$ a normal subgroup then we have the quotient group $(G/N, \bullet, 1_{G/N})$ where G/N is the set of let cosets for N in G, the operation $\bullet$ is given by $gN \bullet g'N = gg'N$, and the identity element $1_{G/N} \in G/N$ is given by left coset of $1_G N = N$. The group G/N is also called sometime "G modulo N".

   (a) Show that the kernel $Ker(pr)$ of the natural map $pr : G \to G/N$, $g \mapsto gN$, satisfies $Ker(pr) = N$.

   (b) Compute the multiplication table for the following quotient groups:

      1. The quotient group $Q = S_4/A_4$ of the symmetric group S_4 modulo the alternating group A_4 of all even elements in S_4.
      2. The quotient group $Q = S_n/A_n$ of the symmetric group S_n modulo the alternating group A_n of all even elements in S_n.
      3. The quotient group $Q = D_3/C_3$ of the dihedral group D_3 of orthogonal symmetries of the triangle modulo the group C_3 of rotational symmetries of the triangle.

(c) Describe a group which is naturally isomorphic to:

1. The quotient group $Q = \mathbb{Z}/4\mathbb{Z}$, where $\mathbb{Z}$ denotes the integers and $4\mathbb{Z} = \{4k; k \in \mathbb{Z}\}$.
2. The quotient group $Q = V/L$, where $V = \mathbb{R}^2$ which is a group with the addition in V, and L is the diagonal subgroup of V, i.e., $L = \{(x, x); x \in \mathbb{R}\}$.
3. The quotient group $Q = GL_2(\mathbb{F}_2)/SL_2(\mathbb{F}_2)$.
4. The quotient group $Q = GL_2(\mathbb{F}_3)/SL_2(\mathbb{F}_3)$.

(d) Suppose G is a group and $H < G$ a subgroup of index $[G : H] = 2$, i.e., $\#(G/H) = 2$. Show that H is normal subgroup, and $G/H \simeq \{1, -1\}$.

3. *Alternating group.* Show that the alternating group A_n is the unique subgroup $H < S_n$ of index two. (Hint: the onto homomorphism $pr : S_n \to S_n/H \simeq \{1, -1\}$, and the homomorphism $sgn : S_n \to \{1, -1\}$ must coincide so their kernels are the same).

4. *Universal property of the quotient group.* Suppose N is a normal subgroup of a group G. Denote by $pr : G \to G/N$ the natural projection (see 2.a. above). Suppose $\varphi : G \to G'$ a homomorphism such that $N < Ker(\varphi)$. Prove the following theorem:

   Theorem. (Universal property of the quotient group). There is a unique homomorphism $\overline{\varphi} : G/N \to G'$ such that (*) $\overline{\varphi} \circ pr = \varphi$.

   Directions: For uniqueness, show that such a $\overline{\varphi}$ must satisfies (**) $\overline{\varphi}(gN) = \varphi(g)$. For existence, show that (**) define a well defined (independent of the representative, i.e., if $gN = g'N$ then $\varphi(g) = \varphi(g')$) homorphism that satisfies (*).

5. *Symmetries of Objects in Space.* Consider in $\mathbb{R}^3$: the Thetrahedron $\mathcal{T}$,the cube $\mathcal{C}$, and the Dodecahedron $\mathcal{D}$. Denote by SO_3 the group of rotations of $\mathbb{R}^3$ (was described in class).

   (a) Describe a natural isomorphism r from the group Thetrahedral group $T = Stab_{SO_3}(\mathcal{T})$ (the rotational symmetries of $\mathcal{T}$) to A_4 (the alternating group of 4 letters). Prove that your map is indeed an isomorphism.

   (b) Describe a natural isomorphism r from the group Octahedral group $O = Stab_{SO_3}(\mathcal{C})$ (the rotational symmetries of $\mathcal{C}$) to S_4 (the permutation group of 4 letters). Prove that your map is indeed an isomorphism.

   (c) Describe a natural isomorphism r from the Dodecahedral group $D = Stab_{SO_3}(\mathcal{D})$ (the rotational symmetries of $\mathcal{D}$) to A_5 (the alternating group of 5 letters). Prove that your map is indeed an isomorphism.

Remarks

- You are very much encouraged to work with other students. However, submit your work alone.
- We will be happy to help you with the home works. Please visit me in my office hours.

Good Luck!