Math 541 Fall 2010
Homework#6, November 10—Homomorphisms, Kernels, Lagrange's Theorem, Orbits

Remark. Answers should be written in the following format:
i) Statement and/or Result.
ii) Main points that will appear in your explanation or proof or computation.
iii) The actual explanation or proof or computation.

1. Let $\varphi : G \to H$ be homomorphism of finite groups.
   (a) Show that $[G : \ker(\varphi)] \cdot \#(\ker(\varphi)) = \#G$ (Hint: This follows directly from Lagrange's theorem).
   (b) Show that $[G : \ker(\varphi)] = \# \operatorname{Im}(\varphi)$ (Hint: we have an isomorphism $\overline{\varphi} : G/\ker(\varphi) \to \operatorname{Im}(\varphi)$).
   (c) Show that $\#G = \#(\ker(\varphi)) \cdot \# \operatorname{Im}(\varphi)$.
2. Suppose G is a finite group acting on a finite set X. For an element $x \in X$ we denote by $G_x = Stab_G(x) = \{g \in G; \ \ g \cdot x = x\}$ the stabilizer subgroup of x.
   (a) Show that G_x is a subgroup of G.
   (b) Let $O \subset X$ be an orbit for the action of G on X. In particular there exist $x \in O$ such that $O = G \cdot x = \{g \cdot x; \, g \in G\}$. Show that $\#O = \#(G_x \backslash G)$.
   (c) Show that the number of elements in any orbit O, for the action of G on X, divides the number of elements in G.
3. Let $P \subset \mathbb{R}^3$ be a regular tetrahedron (symmetric pyramid around the origin). Consider the tetrahedral group $T = Stab_{SO(3)}(P)$, where $SO(3) = \{A \in GL_3(\mathbb{R}); \ \ \langle Au, Av\rangle = \langle u, v\rangle$ for every $u, v \in \mathbb{R}^3$, and $\det(A) = 1\}$. Follow the following steps to show that T is naturally isomorphic to the group A_4 of even permutations of four letters.
   (a) Define the natural map $r : T \to Aut(\{a, b, c, d\})$ and show it is one to one by investigating its kernel.
   (b) Show that the image of r doesn't contains the transpositions. And deduce that $\operatorname{Im}(r) \subset A_4$ (Hint: investigate S_4/A_4 and show that it has order two).
   (c) Show that $\#T \geq 12$ and deduce, using by 3.a and 3.b, that $\#T = 12$. In particular, r is onto and hence isomorphism.

Remarks
- You are very much encouraged to work with other students. However, submit your work alone.
- I will be happy to help you with the homeworks. Please visit me in my office hours.
- Start to think on your project.

Good Luck!