Math 541 Spring 2011
Homework#5, 27/03/11— Orthogonal symmetries of the n-regular polygon, Subgroups of $\mathbb{Z}$, Product, Cyclic groups

Remark. Answers should be written in the following format:
i) Statement and/or Result.
ii) Main points that will appear in your explanation or proof or computation.
iii) The actual explanation or proof or computation.

1. *Orthogonal symmetries of the n-regular polygon.* Consider the set $\mathcal{P}_n \subset \mathbb{R}^2$ with vertices at $\{(\cos(\theta), \sin(\theta));$ where $\theta = k \cdot 360/n,\ k = 0, 1, ..., n-1\}$. Denote by $D_n = Stab_{O(2,\mathbb{R})}(\mathcal{S})$, the group of orthogonal symmetries of $\mathcal{P}_n$. The group D_n is also called the Dihedral group of order n.

    (a) Show that $\#D_n = 2n$, and write explicitly the elements of D_n in term of rotations and reflections with respect to lines.

    1. *Main points.* A linear operator on $\mathbb{R}^2$ is defined unquely by its application on two linearly independent vectors.

    (b) Write also matrices that describe these elements of D_n.

2. *Subgroups of* $\mathbb{Z}$. Consider the group $(\mathbb{Z}, 0, +)$ of integers.

    (a) Show that for every $d \in \mathbb{N} = \{0, 1, 2, ...\}$ the set $d\mathbb{Z} = \{d \cdot n\ ;\ \ n \in \mathbb{Z}\}$ is a subgroup of $\mathbb{Z}$.

    (b) Show the converse, if $H < \mathbb{Z}$ is a subgroup, then there exists an integer $d \geq 0$ such that $H = d\mathbb{Z}$. Hint: Use the Euclid theorem

    $$\text{If } m, n \in \mathbb{N}, \text{ then there exist } q, r \in \mathbb{N} \text{ such that } n = qm + r, \quad \text{with } 0 \leq r < m.$$

3. *Product of groups.* Suppose G and H are groups. Consider the Cartesian product $G \times H$ with its natural multiplication

    $$(g, h) \cdot (g', h') = (g \cdot g', h \cdot h'),$$

    and identity element $(1_G, 1_H)$.

    (a) Consider the sets $\mathbb{Z}_2 = \{0, 1\}$, $\mathbb{Z}_3 = \{0, 1, 2\}$, $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ which are groups with addition $+$ modulo 2, 3, and 6 respectively. Show that the map

    $$\begin{aligned} \varphi \quad &: \quad \mathbb{Z}_6 \to \mathbb{Z}_2 \times \mathbb{Z}_3, \\ x \quad &\mapsto \quad (x \mod 2, \quad x \mod 3), \end{aligned}$$

    is an isomorphism.

4. *Cyclic groups.* A group $(C, \cdot, 1)$ is called <u>cyclic</u> if there exist an element $x \in C$ s.t $C = \{x^k;\ k \in \mathbb{Z}\}$ where

$$x^k = \begin{cases} \underbrace{x \cdot x \cdot \ldots \cdot x}_{k \text{ times}}, & \text{if } k > 0, \\ 1 & \text{if } k = 0, \\ \underbrace{x^{-1} \cdot x^{-1} \cdot \ldots \cdot x^{-1}}_{-k \text{ times}}, & \text{if } k < 0. \end{cases}$$

(a) Show that the group $\mathbb{Z}$ is cyclic.

(b) Show that for every integer $n \geq 1$ the group $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ with operation of addition $+$ modulo n, is cyclic group of order n.

(c) Show that the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not cyclic.

(d) Let C be a cyclic group. Show that if C is of finite order n, i.e., $\#C = n$ for some positive integer n, then C is isomorphic to $\mathbb{Z}_n$. Show that if the cardinality of C is infinite, then C is isomorphic to $\mathbb{Z}$. Hint: Analyze the maps $\mathbb{Z}_n \to C$ given by $k \mapsto x^k$, and $\mathbb{Z} \to C$ given by $x \mapsto x^k$, where x is a generator of C.

Remarks

- You are very much encouraged to work with other students. However, submit your work alone.
- I will be happy to help you with the home works. Please visit me in my office hours.

Good Luck!