Math 541 Spring 2011
Homework#4, 18/02/11— Orthogonal Group, Orthogonal Symmetries of the Triangle and the Square

Remark. Answers should be written in the following format:
i) Statement and/or Result.
ii) Main points that will appear in your explanation or proof or computation.
iii) The actual explanation or proof or computation.

1. *The orthogonal group.* Denote by $\langle\ ,\rangle$ the standard inner product on $\mathbb{R}^2$. Consider the set $O(2,\mathbb{R}) = \{A \in Mat(2,\mathbb{R});$ such that $\langle\ Au, Av\rangle = \langle\ u, v\rangle$ for every $u, v \in \mathbb{R}^2\}$, i.e., the set of all 2×2 real matrices that preserve the inner product.

   (a) Show that for any $A, B \in O(2,\mathbb{R})$, their matrix product $A \circ B \in O(2,\mathbb{R})$.

   (b) Show that the triple $(O(2,\mathbb{R}), \circ, I)$ is a group (subgroup of $GL_2(\mathbb{R})$), where $\circ$ denotes the standard multiplication of matrices, and I is the identity matrix. This group is called the real <u>orthogonal group</u> of order two.

   (c) $(*)$ Show that
   $$O(2,\mathbb{R}) = \{A \in Mat(2,\mathbb{R}); \text{ such that } A \circ A^t = I\}.$$

   (d) Show that if $A \in O(2,\mathbb{R})$ then $\det(A) = \pm 1$. Here, $\det(A)$ is the determinant of A.

   (e) Show that the group R of rotations of the $\mathbb{R}^2$ is equal to the following subgroup of $O(2,\mathbb{R})$
   $$R = \{A \in O(2,\mathbb{R});\ \ \det(A) = 1\}.$$
   In the literature people sometime denote the group R by $SO(2,\mathbb{R})$ and call it the <u>special orthogonal group</u> of order two.

2. *Orthogonal symmetries of the triangle.* Consider the triangle $\mathcal{T} \subset \mathbb{R}^2$ with vertices at $\{(\cos(\theta), \sin(\theta));$ where $\theta = 0°, 120°,$ or $240°\}$.

   (a) Define precisely the orthogonal linear transformations $r_{0,}\ r_{120},\ r_{240}$ of rotational symmetries, and the reflectional symmetries $s_{l_{60}},\ s_{l_{180}},\ s_{l_{300}}$, that preserve $\mathcal{T}$.

   (b) Denote by $D_3 = Stab_{O(2,\mathbb{R})}(\mathcal{T})$ the stabilizer of $\mathcal{T}$ in $O(2,\mathbb{R})$. Explain why $\{r_{0,}\ r_{120},\ r_{240},\ s_{l_{60}},\ s_{l_{180}},\ s_{l_{300}}\} \subset D_3$.

   (c) Show that $\#D_3 = 6$ and deduce that $D_3 = \{r_{0,}\ r_{120},\ r_{240},\ s_{l_{60}},\ s_{l_{180}},\ s_{l_{300}}\}$. In particular, this explains that $\{r_{0,}\ r_{120},\ r_{240},\ s_{l_{60}},\ s_{l_{180}},\ s_{l_{300}}\}$ is a subgroup of $O(2,\mathbb{R})$.

   (d) Compute explicitly the multiplication table of D_3.

3. *Orthogonal symmetries of the square.* Consider the square $\mathcal{S} \subset \mathbb{R}^2$ with vertices at $\{(\cos(\theta), \sin(\theta));$ where $\theta = 0°, 90°,\ 180°,$ or $270°\}$. Denote by $D_4 = Stab_{O(2,\mathbb{R})}(\mathcal{S})$, the group of orthogonal symmetries of $\mathcal{S}$.

(a) Show that $\#D_4 = 8$ and write explicitly the elements of D_4 in term of rotations and reflections.

(b) Write also matrices that describe these elements of D_4.

Remarks

- You are very much encouraged to work with other students. However, submit your work alone.
- I will be happy to help you with the home works. Please visit me in my office hours.

Good Luck!