Math 541 Fall 2010
Homework#4, October 26—Equivalence relations and group action

Remark. Answers should be written in the following format:
i) Statement and/or Result.
ii) Main points that will appear in your explanation or proof or computation.
iii) The actual explanation or proof or computation.

1. A relation on a set X is a subset $\sim\subset X \times X$. If $x, y \in X$ are in relation, i.e., $(x, y) \in\sim$ then we usually write $x \sim y$. A relation $\sim$ on X is called equivalence if it satisfies the following three axioms: i) Reflexivity: if $x \sim x$ for every $x \in X$. ii) Symmetry: If $x \sim y$ then $y \sim x$. iii) Transitivity: If $x \sim y$ and $y \sim z$ then $x \sim z$. An equivalence class for an equivalence relation $\sim$ on X is a maximal subset of X such that all element in it are equivalent. We denote the set of all equivalence classes by $X/\sim$ (it reads: X modulo $\sim$).

   (a) Show that if $\sim$ is an equivalence relation on X then we have a decomposition $X = \bigcup_{E \in X/\sim} E$ into disjoint union over all equivalence classes.

   (b) Suppose we have an action of a group H on a set X. Define the orbit relation on X with $x \sim y$ if x and y are mapped to each other by the action of H. Show that $\sim$ is an equivalence relation. The set $X/\sim$ is denoted in this case by $H\backslash X$ and is called the set of orbits.

   (c) In the following examples describe the orbit set $H\backslash X$:

      1. The natural action of the group $H = SO(2)$ of rotations on $X = \mathbb{R}^2$.
      2. The natural action of the group $H = SO(3)$ on $X = \mathbb{R}^3$.
      3. The natural action of the group $H = \mathbb{R}^*$ on $X = \mathbb{R}^2 - \{0\}$. In this case $H\backslash X$ is denoted by $\mathbb{P}^1$ and is called the projective line.

2. Let H be a subgroup of a finite group G. Then H act on G from the right by by $h \triangleright g = gh^{-1}$. Let us denote the set of equivalence classes for the induced equivalence relation by G/H and by $[G : H]$ the number of elements in G/H. In this case the elements of G/H are called left cosets of H in G.

   (a) Show that any equivalence class in G/H has the same number of elements as in H.

   (b) Deduce the equality of cardinalities $\#G = [G : H] \cdot \#H$. This identity is called Lagrange theorem.

   (c) Let G be a group of order n. Show that for any element $g \in G$ we have $g^n = 1$.

3. Let $C_3 \subset S_3 = Aut(\{a, b, c\})$ be the cyclic group generated by the element $(a\ b\ c)$.

   (a) Show that C_3 is a normal subgroup of S_3.

(b) Describe explicitly the left cosets in S_3/C_3. Do you see a natural group structure on S_3/C_3 such that the natural projection $S_3 \to S_3/C_3$ is an homomorphism.

(c) Find a group P and an homomorphism $\varphi : S_3 \to P$ with kernel C_3.

Remarks

- You are very much encouraged to work with other students. However, submit your work alone.
- I will be happy to help you with the homeworks. Please visit me in my office hours.
- Start to think on your project.

Good Luck!