Math 541 Spring 2011
Homework#3, 14/02/11— Rotational Symmetries, Subgroups of $GL_2(\mathbb{R})$

Remark. Answers should be written in the following format:
i) Statement and/or Result.
ii) Main points that will appear in your explanation or proof or computation.
iii) The actual explanation or proof or computation.

Consider the group $(R, \circ, I)$ of rotations of the plane $\mathbb{R}^2$ of all matrices of the form

$$r_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}, \qquad \theta \in \mathbb{R}.$$

with $\circ$ denotes multiplications of matrices, and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

1. Rotational symmetries of the triangle. Consider the triangle $\mathcal{T}$ with vertices at $(\cos(\theta), \sin(\theta))$, $\theta = 0, 120°, 240°$. Show that if $r_\theta \in R$ preserve $\mathcal{T}$, i.e., $r_\theta(\mathcal{T}) = \mathcal{T}$ then $\theta = 0, 120°, 240°$.

2. Rotational symmetries of the standard n-sided polygon in the plane. Let $n \geq 3$, be an integer. Consider the standard n-sided polygon $\mathcal{P}$ in the plane $\mathbb{R}^2$, with vertices in the points $(\cos(\theta), \sin(\theta))$, $\theta = k \cdot (360/n)$, $k = 0, 1, ..., n-1$. Show that if $r_\theta \in R$ preserve $\mathcal{P}$, i.e., $r_\theta(\mathcal{P}) = \mathcal{P}$, then $\theta = k \cdot (360/n)$, $k = 0, 1, ..., n-1$.

3. Subgroups of $GL_2(\mathbb{R})$. Consider the group $GL_2(\mathbb{R}) = \{A \in Mat(2, \mathbb{R}); A \text{ is invertible}\}$, with the operation $\circ$ of matrix multiplication, and I as the identity element.

   (a) Show that the natural map $GL_2(\mathbb{R}) \times \mathbb{R}^2 \to \mathbb{R}^2$,

   $$(A, \begin{pmatrix} x \\ y \end{pmatrix}) \mapsto A \begin{pmatrix} x \\ y \end{pmatrix},$$

   is an action.

   (b) Consider the collection $P \subset GL_2(\mathbb{R})$ given by

   $$P = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}; \ a, b, c \in \mathbb{R}, \text{ and } a \cdot c \neq 0 \right\}.$$

   Show that $P < GL_2(\mathbb{R})$, i.e., it is a subgroup (it is called the standard parabolic subgroup of $GL_2(\mathbb{R})$). Show this in two ways as follows:

   1. By a direct calculation using the definition of subgroup.
   2. By showing that $P = Stab_{GL_2(\mathbb{R})}(L_1)$, where $L_1 \subset \mathbb{R}^2$ denotes the line

   $$L_1 = \{ \begin{pmatrix} x \\ 0 \end{pmatrix}; \quad x \in \mathbb{R}\}.$$

(c) Consider the collection $U \subset GL_2(\mathbb{R})$ given by

$$P = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} ; \ b \in \mathbb{R} \right\}.$$

Show that $U < GL_2(\mathbb{R})$ a subgroup. It is called the standard unipotent subgroup of $GL_2(\mathbb{R})$.

(d) Consider the collection $SL_2(\mathbb{R}) \subset GL_2(\mathbb{R})$ given by

$$SL_2(\mathbb{R}) = \{A \in GL_2(\mathbb{R}); \ \det(A) = 1\}.$$

Show that $SL_2(\mathbb{R}) < GL_2(\mathbb{R})$ a subgroup. It is called the special linear group of order two over $\mathbb{R}$.

(e) Consider the collection $A \subset GL_2(\mathbb{R})$ given by

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix} ; \ a \cdot c \neq 0 \right\}.$$

Show that $A < GL_2(\mathbb{R})$, i.e., it is a subgroup (it is called the standard diagonal subgroup of $GL_2(\mathbb{R})$). Show this in two ways as follows:

1. By a direct calculation using the definition of subgroup.
2. By showing that $A = Stab_{GL_2(\mathbb{R})}(L_1) \cap Stab_{GL_2(\mathbb{R})}(L_2)$, where $L_1, L_2 \subset \mathbb{R}^2$ denote the lines

$$L_1 = \{ \begin{pmatrix} x \\ 0 \end{pmatrix} ; \quad x \in \mathbb{R}\}, \text{ and } L_2 = \{ \begin{pmatrix} 0 \\ y \end{pmatrix} ; \quad y \in \mathbb{R}\}.$$

   Note, that you should also prove that the intersection of two subgroups is again a subgroup.

Remarks

- You are very much encouraged to work with other students. However, submit your work alone.
- I will be happy to help you with the home works. Please visit me in my office hours.

Good Luck!