Math 541 - Fall 2017
HW3 - Group Actions, Stabilizers, Subgroups
For the Friday 10/6/17 "10 min." test

Remark. Answers should be written in the following format:
i) Statement and/or Result.
ii) Main points that will appear in your explanation or proof or computation.
iii) The actual explanation or proof or computation.

1. *Action of a group on a set.* Lets X be a set and $(G, *.1_G)$ a group.

   (a) Define what we mean when we say that G acts on X. We also say in this case that X is a G-set.

   (b) Denote by $Aut(X) = \{\sigma : X \to X \mid \text{s.t. } \sigma \text{ is bijection}\}$. Consider the group $(Aut(X), \circ, id)$, where $\circ$ denotes composition of functions, and id the identity map on X. Show that the following are equivalent:

      1. X is a G-set.
      2. There is a map $\alpha : G \to Aut(X)$ s.t. $\alpha(g * g') = \alpha(g) \circ \alpha(g')$, for every $g, g' \in G$.

   (c) Show that there is a G-set X such that the natural map α you defined in 2) above is injection (i.e., one-to-one).

2. *Isomorphism and Cayley theorem. Suppose* $(G, *_G, 1_G)$ and $(H, *_H, 1_H)$ are groups. A map $\alpha : G \to H$ is called isomorphism if (i) It is homomorphism, i.e., for every $g, g' \in G$, $\alpha(g *_G g') = \alpha(g) *_H \alpha(g')$ and (ii) It is bijection. If there is such isomorphism between G and H, then we say that G and H are isomorphic, denoted $G \simeq H$.

   (a) Show that if $\alpha : G \to G'$ is homomorphism between groups, then:

      1. $\alpha(1_G) = 1_{G'}$.
      2. The image of α, $\mathrm{Im}(\alpha) = \{\alpha(g);\ g \in G\}$ is a subgroup of G'.
      3. If α is one-to-one (i.e., for $g \neq g'$ in G, we have $\alpha(g) \neq \alpha(g')$), then the induced map $\overline{\alpha} : G \to \mathrm{Im}(\alpha)$ is an isomorphism.

   (b) A collection H is called group of transformations of a set X, if H is a subgroup of $Aut(X)$. Prove the following:

      Theorem (Cayley) Every group is isomorphic to a group of transformations of some set X.

3. *The group O_2.* Let $(G, *, 1_G)$ be a group acting on a set X. For a subset $S \subset X$ we define the set of *G-symmetries* of S to be the subset $Stab_G(S) \subset G$ given by

$$Stab_G(S) = \{g \in G;\ g(S) = S\},$$

   where $g(S) = \{g \cdot s;\ s \in S\}$. The set $Stab_G(S)$ is also called the stabilizer of S in G.

   (a) Show that $Stab_G(S)$ is a subgroup of G.

(b) Consider the collection

$$GL_2(\mathbb{R}) = \{A \in M_2(\mathbb{R});\ A \text{ is invertible linear transformation}\},$$

where $M_2(\mathbb{R})$ is the collection of 2×2 matrices with real entries. Show that $(GL_2(\mathbb{R}), *, I_2)$ is a group, with $*$ matrix multiplications, and I_2 the 2×2 identity matrix.

Definition. A map $B : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is called bilinear form on $\mathbb{R}^2$, if it is linear in each coordinate, i.e., $B(u+v, w) = B(u, w)+B(v, w)$ and $B(au, w) = aB(u, w)$ for every $u, v, w \in \mathbb{R}^2$ and $a \in \mathbb{R}$, and likewise in the second coordinate.

(c) Let us denote by $Bil(\mathbb{R}^2)$ the collection of all bilinear forms on $\mathbb{R}^2$. Show that $GL_2(\mathbb{R})$ acts on $Bil(\mathbb{R}^2)$ using the formula

$$[g \cdot B](u, v) = B(g^{-1}u, g^{-1}v),$$

for every $g \in GL_2(\mathbb{R})$, $B \in Bil(\mathbb{R})$, and $u, v \in \mathbb{R}^2$. Namely, prove that the map $\cdot : GL_2(\mathbb{R}) \times Bil(\mathbb{R}^2) \to Bil(\mathbb{R}^2)$, given by $(g, B) \mapsto g \cdot B$, is an action.

(d) An important example of bilinear form on $\mathbb{R}^2$ is the standard inner product $\langle\ ,\rangle$. Consider the subset $\{\langle\ ,\rangle\} \subset Bil(\mathbb{R}^2)$, and the collection

$$O_2 = Stab_{GL_2(\mathbb{R})}(\{\langle\ ,\rangle\}).$$

Show that O_2 is a subgroup of $GL_2(\mathbb{R})$. It is called the orthogonal group.

4. The group S_n. Fix an integer $n \geq 1$, and consider the set $\{1, ..., n\}$. The group $S_n = Aut(\{1, ..., n\})$ is called the symmetric group on n symbols, or the group of permutations of n symbols.

(a) Compute the cardinality $\#S_n$.

(b) A common way to write a permutation $\sigma \in S_n$, is to use a table

$$\begin{pmatrix} 1 & \dots & i & \dots & n \\ \sigma(1) & \dots & \sigma(i) & \dots & \sigma(n) \end{pmatrix},$$

describing the image to which an element $i = 1, .., n$ is sent by σ. Write all the elements of S_4.

(c) Consider the group $S_3 = Aut(\{1, 2, 3\})$.

1. For each k that divides $\#S_3$ find a subgroup $H_k < S_3$ with cardinality $\#H_k = k$.
2. For each k as above find some S_3-set X_k and some subset $Y_k \subset X_k$ such that $H_k = Stab_{S_3}(Y_k)$.

Remarks

- You are very much encouraged to work with other students on the HW.
- We encourage you at attend the **Discussion meeting every Wed. 4-5pm at B333.**
- We will be happy to help you with the home works. Please visit us in our office hours.

Good Luck!