Math 541 Fall 2010
Homework#3, October 25—Isomorphisms and some ways to get them

Remark. Answers should be written in the following format:
i) Statement and/or Result.
ii) Main points that will appear in your explanation or proof or computation.
iii) The actual explanation or proof or computation.

1. Let G be a group.

   (a) Consider the set $Aut(G) = \{\varphi : G \to G;\ \ \varphi$ is an isomorphism of groups$\}$. Show that $Aut(G)$ is a group with respect to the operation of composition of maps, and with identity the homomorphism $id : G \to G$, $id(g) = g$ for every $g \in G$.

   (b) For an element $g \in G$ consider the map $Ad(g) : G \to G$ given by $Ad(g)(x) = gxg^{-1}$. Show that $Ad(g)$ is an isomorphism.

2. We say that two subgroups K and H of a group G are conjugated one to the other if there exist an element $g \in G$ such that $K = H^g$ where $H^g = gHg^{-1} = \{ghg^{-1};\ h \in H\}$. It is clear that in this case $Ad(g)$ is an isomorphism between H and K.

   (a) Suppose we have an action of a group G on a set X. For a subset $Y \subset X$ and an element $g \in G$ show that the subgroups $H = Stab_G(Y)$ and $K = Stab_G(g(Y))$ are conjugated.

   (b) Consider a triangle $\triangle$ in the plane and a rotation R_θ of the plane by angle θ around the origin. Show that the subgroups $H = Stab_{GL_2(\mathbb{R})}(\triangle)$ and $K = Stab_{GL_2(\mathbb{R})}(R_\theta(\triangle))$ are conjugated.

3. Let C be a symmetric cube around the origin in $\mathbb{R}^3$. Let $G = SO(3) = \{A \in GL_3(\mathbb{R});\ AA^t = I$ and $\det(A) = 1\}$ the group of rotations of the space around the origin. Show that there exist a natural isomorphism $r : Stab_G(C) \xrightarrow{\sim} S_4$. Clue: C has four diagonals.

Remarks

- You are very much encouraged to work with other students. However, submit your work alone.
- I will be happy to help you with the homeworks. Please visit me in my office hours.
- Start to think on your project.

Good Luck!