Math 541 Spring 2011
Homework#3—Answers— Rotational Symmetries, Subgroups of $GL_2(\mathbb{R})$

Consider the group $(R, \circ, I)$ of rotations of the plane $\mathbb{R}^2$ of all matrices of the form

$$r_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}, \qquad \theta \in \mathbb{R}.$$

with $\circ$ denotes multiplications of matrices, and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

1. Rotational symmetries of the triangle. Consider the triangle $\mathcal{T}$ with vertices at $(\cos(\theta), \sin(\theta))$, $\theta = 0, 120°, 240°$. Show that if $r_\theta \in R$ preserve $\mathcal{T}$, i.e., $r_\theta(\mathcal{T}) = \mathcal{T}$ then $\theta = 0, 120°, 240°$.

   (a) *Main points.* Direct calculation.

   (b) *Proof.* If $r_\theta(\mathcal{T}) = \mathcal{T}$ then it should takes the vector $(1, 0)$ to $(\cos(\theta), \sin(\theta))$, where $\theta = 0, 120, 240$.

2. Rotational symmetries of the standard n-sided polygon in the plane. Let $n \geq 3$, be an integer. Consider the standard n-sided polygon $\mathcal{P}$ in the plane $\mathbb{R}^2$, with vertices in the points $(\cos(\theta), \sin(\theta))$, $\theta = k \cdot (360/n)$, $k = 0, 1, ..., n-1$. Show that if $r_\theta \in R$ preserve $\mathcal{P}$, i.e., $r_\theta(\mathcal{P}) = \mathcal{P}$, then $\theta = k \cdot (360/n)$, $k = 0, 1, ..., n-1$.

   (a) *Main points.* Direct calculation.

   (b) *Proof.* If $r_\theta(\mathcal{P}) = \mathcal{P}$ then it should takes the vector $(1, 0)$ to $(\cos(\theta), \sin(\theta))$, where $\theta = k \cdot (360/n)$, $k = 0, 1, ..., n-1$.

3. Subgroups of $GL_2(\mathbb{R})$. Consider the group $GL_2(\mathbb{R}) = \{A \in Mat(2, \mathbb{R}); A \text{ is invertible}\}$, with the operation $\circ$ of matrix multiplication, and I as the identity element.

   (a) Show that the natural map $GL_2(\mathbb{R}) \times \mathbb{R}^2 \to \mathbb{R}^2$,

   $$(A, \begin{pmatrix} x \\ y \end{pmatrix}) \mapsto A \begin{pmatrix} x \\ y \end{pmatrix},$$

   is an action.

   1. *Main points.* Definition of action, and properties of matrix multiplication.
   2. *Proof.*
      1. *Associativity.* Follows from the fact that matrix multiplication is associative.
      2. *Identity.* $I \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$.

(b) Consider the collection $P \subset GL_2(\mathbb{R})$ given by

$$P = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} ; \ a, b, d \in \mathbb{R}, \text{ and } a \cdot d \neq 0 \right\}.$$

Show that $P < GL_2(\mathbb{R})$, i.e., it is a subgroup (it is called the standard parabolic subgroup of $GL_2(\mathbb{R})$). Show this in two ways as follows:

1. By a direct calculation using the definition of subgroup.
   1. *Main points.* Characterization of subgroup.
   2. *Proof.* (1) $P \neq \varnothing$. (2) Let us show that that P is close under the operation of matrix multiplication. Indeed,

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \cdot \begin{pmatrix} a' & b' \\ 0 & d' \end{pmatrix} = \begin{pmatrix} aa' & ab' + bd' \\ 0 & dd' \end{pmatrix}$$

   and $(aa')(dd') \neq 0$. And finally (3)

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^{-1} = \begin{pmatrix} a^{-1} & -\frac{b}{ad} \\ & d^{-1} \end{pmatrix}$$

   which is in P.
2. By showing that $P = Stab_{GL_2(\mathbb{R})}(L_1)$, where $L_1 \subset \mathbb{R}^2$ denotes the line

$$L_1 = \{ \begin{pmatrix} x \\ 0 \end{pmatrix} ; \quad x \in \mathbb{R} \}.$$

   1. *Main points.* Direct calculation.
   2. *Calculation.* We need to find all the invertible matrices $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ that satisfies for every $x \in \mathbb{R}$

$$\begin{pmatrix} ax \\ cx \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} x' \\ 0 \end{pmatrix},$$

   for some $x' \in \mathbb{R}$. We deduce that $c = 0$, and this is the only constraint.

(c) Consider the collection $U \subset GL_2(\mathbb{R})$ given by

$$U = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} ; \ b \in \mathbb{R} \right\}.$$

Show that $U < GL_2(\mathbb{R})$ a subgroup. It is called the standard unipotent subgroup of $GL_2(\mathbb{R})$.

1. *Main points.* Stabilizer or direct calculation.
2. 1. *Proof by stabilizer method.* Consider the group $SL_2(\mathbb{R})$ defined in the next section. Then

$$U = Stab_{SL_2(\mathbb{R})}(\begin{pmatrix} 1 \\ 0 \end{pmatrix}).$$

Indeed, if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, i.e., $\det(A) = 1$ and if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

then $a = 1,\ c = 0,\ d = 1$.

2. *Proof by direct calculation. We have*

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & b' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b+b' \\ 0 & 1 \end{pmatrix},$$

which proofs the closure part, and

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix},$$

which proofs the closure to inverse.

(d) Consider the collection $SL_2(\mathbb{R}) \subset GL_2(\mathbb{R})$ given by

$$SL_2(\mathbb{R}) = \{A \in GL_2(\mathbb{R});\ \det(A) = 1\}.$$

Show that $SL_2(\mathbb{R}) < GL_2(\mathbb{R})$ a subgroup. It is called the special linear group of order two over $\mathbb{R}$.

1. *Main points.* Property of determinant.
2. Proof. Let $A, B \in SL_2(\mathbb{R})$ then $\det(A \cdot B) = \det(A)\det(B) = 1 \cdot 1 = 1$, so $A \cdot B \in SL_2(\mathbb{R})$. Also, $\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{1} = 1$, so $A^{-1} \in SL_2(\mathbb{R})$.

(e) Consider the collection $A \subset GL_2(\mathbb{R})$ given by

$$A = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix};\ a \cdot d \neq 0 \right\}.$$

Show that $A < GL_2(\mathbb{R})$, i.e., it is a subgroup (it is called the standard diagonal subgroup of $GL_2(\mathbb{R})$). Show this in two ways as follows:

1. By a direct calculation using the definition of subgroup.
   1. Indeed,

$$\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \cdot \begin{pmatrix} a' & 0 \\ 0 & d' \end{pmatrix} = \begin{pmatrix} aa' & 0 \\ 0 & dd' \end{pmatrix},$$

which proofs closure, and

$$\begin{pmatrix} a & \\ & d \end{pmatrix}^{-1} = \begin{pmatrix} a^{-1} & \\ & d^{-1} \end{pmatrix},$$

which proofs closure to inverse.

2. By showing that $A = Stab_{GL_2(\mathbb{R})}(L_1) \cap Stab_{GL_2(\mathbb{R})}(L_2)$, where $L_1, L_2 \subset \mathbb{R}^2$ denote the lines

$$L_1 = \{\begin{pmatrix} x \\ 0 \end{pmatrix}; \quad x \in \mathbb{R}\}, \text{ and } L_2 = \{\begin{pmatrix} 0 \\ y \end{pmatrix}; \quad y \in \mathbb{R}\}.$$

Note, that you should also prove that the intersection of two subgroups is again a subgroup.

1. Indeed, If $A \in GL_2(\mathbb{R})$ preserves L_1 then as we calculated before it is of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ with $a \cdot d \neq 0$. In the same way, if it preserves L_2 then it is of the form $\begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$. Now, if G_1, G_2 are subgroups of G, and if $g, g' \in G_1 \cap G_2$ then $g \cdot g'$ are in both G_1 and G_2. Moreover, g^{-1} is also both in G_1 and G_2. Of course $1 \in G_1 \cap G_2$. So, we conclude that $G_1 \cap G_2 < G$.

Remarks • I will be happy to go with you over these answers. Please visit me in my office hours.

Good Luck!