Math 541 Spring 2011
Homework#2, 01/02/11— Rotational symmetries

Remark. Answers should be written in the following format:
i) Statement and/or Result.
ii) Main points that will appear in your explanation or proof or computation.
iii) The actual explanation or proof or computation.

1. Rotational symmetries.

   (a) Consider the collection R of all matrices of the form

   $$r_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}, \qquad \theta \in \mathbb{R}.$$

   Show that the triple $(R, \circ, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})$ with $\circ$ denotes multiplications of matrices, is a group. This group is called the *group of rotations* of the plane $\mathbb{R}^2$. In the literature it is also denoted sometime by $SO(2, \mathbb{R})$.

   (b) Write down the definition of a group $(G, \cdot, 1)$ acting on a set X.

   (c) Consider the natural map $R \times \mathbb{R}^2 \to \mathbb{R}^2$, which sends a pair (r_θ, v) to the vector $r_\theta(v)$, i.e., the rotation of v by angle θ. Show that this map satisfies the axioms for the action of a group on a set.

2. Rotational symmetries of a square.

   (a) Write down a diagram of the standard square $\square \subset \mathbb{R}^2$ with vertices $(\pm 1, \pm 1)$.

   (b) Compute the set $C_4 = \{r_\theta \in R;\ r_\theta(\square) = \square\}$ of all rotational symmetries of the square.

   (c) Show that the triple $(C_4, \circ, I)$, where $\circ$ denotes the operation of composition of rotations, and I is the identity rotation, is a group. We will call it the *group of rotational symmetries of the square.*

3. Let $(G, \cdot, 1)$ be a group acting on a set X. For a subset $Y \subset X$ we define *the set of G-symmetries of Y* to be the subset $Stab_G(Y) \subset G$ given by

   $$Stab_G(Y) = \{g \in G;\ g(Y) = Y\},$$

   where $g(Y) = \{g \cdot y;\ y \in Y\}$. The set $Stab_G(Y)$ is also called the *stabilizer* of Y in G.

   (a) Show that $Stab_G(Y)$ is a subgroup of G.

   (b) Consider the standard 5-sided polygon Y in the plane $\mathbb{R}^2$, with vertices in the points $z^5 = 1$, where we think on points of $\mathbb{R}^2$ as complex numbers $z \in \mathbb{C}$. Write down its diagram.

(c) Compute the group $C_5 = Stab_R(Y) = \{r_\theta \in R;\, r_\theta(Y) = Y\}$ of rotational symmetries of Y.

Remarks

- You are very much encouraged to work with other students. However, submit your work alone.
- I will be happy to help you with the home works. Please visit me in my office hours.

Good Luck!