Math 541 Fall 2010
Homework#2, Sep 21—Groups, Group Action ("Symmetry"), Isomorphism

Remark. Answers should be written in the following format:
i) Statement and/or Result.
ii) Main points that will appear in your explanation or proof or computation.
iii) The actual explanation or proof or computation.

1. (Products of two groups). Let $(G, *, 1_G)$ and $(H, \cdot, 1_H)$ be two groups. We define a new group, called the product of G and H, by simply considering the set $G \times H = \{(g, h);$ $g \in G$ and $h \in H\}$ with the multiplication law $\circ$ given by $(g, h) \circ (g', h') = (g * g, , h \cdot h')$ and the identity element $(1_G, 1_H)$.

   (a) What is the inverse of an element (g, h)? (clue: g^{-1} is the inverse of g).

   (b) Let $(\mathbb{Z}_2, +, 0)$ be the group with $\mathbb{Z}_2 = \{0, 1\}$ and $+$ is addition of integers modulo 2. Write the multiplication table (see HW #1) of the group $\mathbb{Z}_2 \times \mathbb{Z}_2$.

   (c) Write the multiplication table for the group $(\mathbb{Z}_4, +, 0)$.

   (d) Recall that a set X is called G-set if there exist and action of G on X. Find a set X which is a G-set for $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and a set Y which is a H-set for $H = \mathbb{Z}_4$ (clue: you can take the square in the plane, but write down your action).

2. (Order of an element in a group). Let G be a group and $g \in G$. The order of g, denoted by $ord(g)$, is the least integer $n > 0$ such that $g^n = 1_G$, where g^n is the multiplication n times $g \cdot g \cdot .... \cdot g$. If there is no such n then we say that the order of g is infinite and write $ord(g) = \infty$.

   (a) Show that if G has finitely many elements (we write also $\#G < \infty$ and such a group is also called finite) then every element is of finite order.

   (b) Compute the order of the element 1 in $\mathbb{Z}_4$. Compute the order of any element $(0, 0) \neq (x, y) \in \mathbb{Z}_2 \times \mathbb{Z}_2$.

   (c) We say that two groups $(G, *, 1_G)$ and $(H, \cdot, 1_H)$ are isomorphic if there exist a bijection $\varphi : G \to H$ that satisfies $\varphi(g_1 * g_2) = \varphi(g_1) \cdot \varphi(g_2)$ (i.e. "it takes product to product"). Show that the groups $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ are not isomorphic (clue: you can use subsection b. above).

3. Group action and symmetry.

   (a) Let $\triangle$ be an equilateral triangle in the plane $\mathbb{R}^2$ with center at $(0, 0)$. Denote by T the subgroup of $GL_2(\mathbb{R})$ that takes T to itself. i.e., $T = \{A \in GL_2(\mathbb{R});$ A takes vertices of $\triangle$ to vertices of $\triangle$ and corresponding edges to edges$\}$. Compute explicitly the group T (Clue: some rotations and some reflections). Conclude that $\#T = 6$.

(b) Denote by $S_3 = Aut(\{a,b,c\})$ the group of all bijections from the set $\{a,b,c\}$ to itself with the standard operation of composition of maps. Compute explicitly all the elements of S_3. Conclude that $\#S_3 = 6$.

(c) Show that the group T and S_3 are naturally isomorphic. How to do that? Here I will help you so please follow the following steps: Denote the vertices of $\triangle$ by a,b and c. Consider the map $\varphi : T \to S_3$ that sends the matrix $A \in T$ to the following element of S_3

$$\begin{pmatrix} a & b & c \\ Aa & Ab & Ac \end{pmatrix}.$$

1. Show that $\varphi(A \cdot B) = \varphi(A) \circ (B)$ i.e. it send composition of matrices from T to compositions of bijections in S_3.
2. Show that φ is one-to-one.
3. Conclude that φ is onto.

4. Define on the collection of all groups a relation $\simeq$ as follows: $G \simeq H$ if they are isomorphic. Show that $\simeq$ is an equivalence relation. i.e.,

(a) We have $G \simeq G$ for every group G.

(b) If $G \simeq H$ then $H \simeq G$.

(c) If $G \simeq H$ and $H \simeq K$ then $G \simeq K$.

Remarks

- You are very much encouraged to work with other students. However, submit your work alone.
- I will be happy to help you with the homeworks. Please visit me in my office hours.

Good Luck!