Math 541 Spring 2011
Homework#1, 25/01/11— Notion of a group and some examples

Remark. Answers should be written in the following format:
i) Statement and/or Result.
ii) Main points that will appear in your explanation or proof or computation.
iii) The actual explanation or proof or computation.

Definition. A group is a triple $(G, \cdot, 1_G)$, where G is a set, $\cdot : G \times G \to G$, $(g, h) \mapsto g \cdot h$, is a map, called *operation,* and $1_G \in G$ a specific element, called *identity*, such that the following axioms are satisfied:

- Associativity. We have $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$ for every $g_1, g_2, g_3 \in G$.
- Identity. The element 1_G satisfies $1_G \cdot g = g \cdot 1_G = g$ for every $g \in G$.
- Inverse. For every $g \in G$ there exists $g' \in G$ such that $g \cdot g' = g' \cdot g = 1_G$. We will denote such a g' (it turns out that it is unique see 2.b. below) usually by g^{-1}.

1. Check which of the following is a group

   (a) The set $\mathbb{R}$ with the standard operation of addition $+$ and identity element 0.

   (b) The set $\mathbb{N} = \{0, 1, 2, ...\}$ of natural numbers, with the standard operation of addition and the element 0 as identity.

   (c) The set $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ of integers, with the standard operation of addition and the element 0 as identity.

   (d) The triple $(\mathbb{R}^\times, \cdot, 1)$ where $\mathbb{R}^\times = \mathbb{R} - 0$ the set of all non-zero real numbers, $\cdot$ is the usual multiplication of real numbers, and 1 is the usual number 1.

   (e) Denote by $GL_2(\mathbb{R})$ the set of all 2×2 invertible matrices with real entries

   $$GL_2(\mathbb{R}) = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} ;\ a, b, c, d \in \mathbb{R} \text{ and there exists } A^{-1} \right\}.$$

   Show that with the operation $\circ$ of matrix multiplication the triple $(GL_2(\mathbb{R}), \circ, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix})$ is a group.

   (f) The triple $(M_2(\mathbb{R}), +, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix})$ where $M_2(\mathbb{R})$ denotes the set of all 2×2 matrices with real entries, and $+$ denotes standard addition of matrices.

2. Properties of groups. Let $(G, \cdot, 1)$ be a group. Show that:

   (a) If e and e' are elements of G which satisfy $e \cdot g = g \cdot e = g$ and $e' \cdot g = g \cdot e' = g$ then $e = e'$.

(b) If g,g', g'' are elements of G, and $g \cdot g' = g' \cdot g = 1$ and $g'' \cdot g = g \cdot g'' = 1$ then $g' = g''$.

(c) If $g, g' \in G$ and $g \cdot g' = 1$ then $g' \cdot g = 1$.

(d) For every $g \in G$ we have $(g^{-1})^{-1} = g$.

(e) For every $g, h \in G$ we have $(g \cdot h)^{-1} = h^{-1} \cdot g^{-1}$.

3. Permutations. Let X be the set $X = \{a, b, c\}$. A function $\sigma : X \to X$ is called *bijection*, or *permutation*, or *isomorphism*, or *automorphism* if it is: (A) One-to-one, also denoted $1 - 1$, i.e., σ satisfies the property that $\sigma(x) = \sigma(y)$ implies $x = y$ for every $x, y \in X$,and (B) Onto, i.e., σ satisfies the property that for every $y \in X$ there exists $x \in X$ with $\sigma(x) = y$.

   (a) Denote by $Aut(X) = \{\sigma : X \to X \, ; \sigma \text{ is a bijection}\}$ the set of ALL bijections from X to itself. Show that with the operation of standard composition $\circ$ of functions, and the element $id : X \to X$, $id(x) = x$ for every $x \in X$, we have that the triple $(Aut(X), \circ, id)$ is a group. The group $Aut(\{a, b, c\})$ is also denoted by S_3 and is called sometime the symmetric group on three letters.

   (b) Show that the number of elements in S_3 is 6.

   (c) Write all the elements of S_3.

   (d) A group G is called *commutative* if $g \cdot h = h \cdot g$ for every $g, h \in G$. Show that the group $(\mathbb{Z}, +, 0)$ of integers is commutative. Show that the group S_3 is not commutative.

Remarks

- You are very much encouraged to work with other students. However, submit your work alone.
- I will be happy to help you with the home works. Please visit me in my office hours.

Good Luck!