Math 541 - Fall 2017
HW1 - Motivations, Notion of a Group, Examples
For the 9/15/17 Friday's "10min" test

Remark. Answers should be written in the following format:
i) Statement and/or Result.
ii) Main points that will appear in your explanation or proof or computation.
iii) The actual explanation or proof or computation.

1. *Symmetries of the n-gon.*

   (a) Define the n-gons $\mathcal{G}_n$ around the origin in the plane $\mathbb{R}^2$. Draw $\mathcal{G}_n$ for $n = 3, 4, 5$, called *triangle, square, and pentagon, respectively.*

   (b) For a vector space V over a field $\mathbb{F}$, define when we say that a map $A : V \to V$ is a <u>linear transformation</u>.

   (c) Compute **all** (and explain why these are all!) the linear transformations $A : \mathbb{R}^2 \to \mathbb{R}^2$, that preserve:

      1. The triangle $\mathcal{T} = \mathcal{G}_3$.
      2. The square $\mathcal{S} = \mathcal{G}_4$.
      3. The pentagon $\mathcal{P} = \mathcal{G}_5$.
      4. The general n-gon $\mathcal{G}_n$.

2. *Definition and first examples of groups.*

   Definition. A <u>group</u> is a triple $(G, \cdot, 1_G)$, where G is a set, $\cdot : G \times G \to G$, $(g, h) \mapsto g \cdot h$, is a map, called <u>operation,</u> and $1_G \in G$ a specific element, called <u>identity</u>, such that the following axioms are satisfied:

   - *Associativity.* We have $g_1 \cdot (g_2 \cdot g_3) = (g_1 \cdot g_2) \cdot g_3$ for every $g_1, g_2, g_3 \in G$.
   - *Identity.* The element 1_G satisfies $1_G \cdot g = g \cdot 1_G = g$ for every $g \in G$.
   - *Inverse.* For every $g \in G$ there exists $g' \in G$ such that $g \cdot g' = g' \cdot g = 1_G$. We will denote such a g' (it turns out that there is only one such element) usually by g^{-1}.

     Notation. For a vector space V we denote by $\circ$ the operation of <u>composition</u> of linear transformation from V to itself, given by $(A \circ B)(v) = A(B(v))$, for every two linear transformations A, B on V, and any vector $v \in V$. In addition, we denote by Id_V the <u>identity</u> linear transformation on V, given by $Id_V(v) = v$ for every $v \in V$.

   (a) Let us denote by T the collection of all linear transformation on $\mathbb{R}^2$ that preserve the triangle $\mathcal{T} = \mathcal{G}_3$. Show that the triple $(T, \circ, 1_T = Id_{\mathbb{R}^2})$ is a group. Write down the <u>multiplication table</u> for the group T.

   (b) Let us denote by S the collection of all linear transformation on $\mathbb{R}^2$ that preserve the square $\mathcal{S} = \mathcal{G}_4$. Show that the triple $(S, \circ, 1_S = Id_{\mathbb{R}^2})$ is a group.

(c) Let us denote by P the collection of all linear transformation on $\mathbb{R}^2$ that preserve the pentagon $\mathcal{P} = \mathcal{G}_5$. Show that the triple $(P, \circ, 1_P = Id_{\mathbb{R}^2})$ is a group.

3. *Permutations.*

Definition. Let X be a set. A function $\sigma : X \to X$ is called bijection, or permutation, or isomorphism, or automorphism if it is:

- One-to-one, also denoted $1-1$, i.e., σ satisfies the property that $\sigma(x) = \sigma(y)$ implies $x = y$ for every $x, y \in X$.
- Onto, i.e., σ satisfies the property that for every $y \in X$ there exists $x \in X$ with $\sigma(x) = y$.

Denote by $Aut(X) = \{\sigma : X \to X \ ; \ \sigma \text{ is a bijection}\}$ the set of ALL bijections from X to itself.

(a) Show that with the operation of standard composition $\circ$ of functions, and the element $id : X \to X$, $id(x) = x$ for every $x \in X$, we have that the triple $(Aut(X), \circ, id)$ is a group.

(b) The group $Aut(\{a, b, c\})$ is also denoted by S_3 and is called sometime the symmetric group on three letters.

1. Show that the number of elements in S_3 is 6.
2. Write all the elements of S_3.
3. Write down the multiplication table for the group S_3.

- You are very much encouraged to work with other students on the HW.
- We encourage you at attend the **Discussion meeting every Wed. 4-5pm at B333.**
- We will be happy to help you with the home works. Please visit us in our office hours.

Good Luck!