Math 541 Fall 2010
Homework#1, Sep 15—Around the definition of a group

Remark. Answers should be written in the following format:
i) Statement and/or Result.
ii) Main points that will appear in your explanation or proof or computation.
iii) The actual explanation or proof or computation.

The following has nothing to do with the definition of a group but it is one of the central theorems of linear algebra. I would like you to read its proof for example as it appears in the book of I. M. Gelfand, Lectures on Linear Algebra page 107–108. There is no need to submit it.
Theorem. Let V be a finite dimensional vector space over the field of complex numbers $\mathbb{C}$. Assume that $T, S : V \to V$ two diagonalizable linear transformations which commutes. i.e., $T \circ S = S \circ T$. Then there exist a basis $\mathcal{B}$ for V consisting of common eigenvectors of T and S.

Now for the exercises.

1. Check which of the following is a group.
   (a) $(\mathbb{R}^{\times}, \cdot, 1)$ where $\mathbb{R}^{\times}$ is the set of non-zero real numbers, $\cdot$ is the usual multiplication of real numbers.
   (b) $(\mathbb{R}, +, 0)$.
   (c) $(GL_2(\mathbb{R}), \cdot, I_2)$ where $GL_2(\mathbb{R})$ the set of all 2×2 matrices with non-zero determinant, $\cdot$ the operation of matrix multiplication and I_2 the identity matrix.
   (d) $(SL_2(\mathbb{R}), \cdot, I_2)$ where $SL_2(\mathbb{R})$ the set of all 2×2 matrices with determinant equal to one, $\cdot$ the operation of matrix multiplication and I_2 the identity matrix.
   (e) $(M_2(\mathbb{R}), \cdot, I_2)$ where $M_2(\mathbb{R})$ the set of all 2×2 matrices and $\cdot$ the operation of matrix multiplication.
   (f) $(Aut(X), \circ, Id)$ where for a set X we denote by $Aut(X) = \{\sigma : X \to X;\ \sigma$ is a function which is one-to-one and onto$\}$, $\circ$ composition of functions and $Id : X \to X$ is the identity function $Id(x) = x$ for every $x \in X$.
   (g) $(\mathbb{Z}, +, 0)$ where $\mathbb{Z}$ the set of all integers.

2. Let $(G, \cdot, 1)$ be a group. Show:
   (a) $(g^{-1})^{-1} = g$ for every $g \in G$.
   (b) $(g \cdot h)^{-1} = h^{-1} \cdot g^{-1}$ for every $g, h \in G$.

3. A group G is called commutative (or Abelian) if for every $g, h \in G$ we have $g \cdot h = h \cdot g$.
   (a) Give an example of a commutative group.

(b) Show that $GL_2(\mathbb{R})$ is not commutative.

4. Let G be a group. There is a way to recall the multiplication law of G as follow. Assume that G is finite (i.e., has finitely many elements) write $G = \{g_1 = 1, g_2, ..., g_n\}$. The table

$$M(\ G) = \begin{array}{ccccc} \cdot & 1 & g_2 & \cdots\cdots & g_n \\ 1 & 1 & g_2 & \cdots\cdots & g_n \\ g_2 & g_2 & g_2 \cdot g_2 & \cdots\cdots & g_2 \cdot g_n \\ \vdots & \vdots & \vdots & \cdots\cdots & \vdots \\ g_n & g_n & g_n \cdot g_2 & \cdots\cdots & g_n \cdot g_n \end{array}$$

is called the multiplication table of G.

(a) Write the multiplication table of $S_3 = Aut(X)$ where $X = \{a, b, c\}$.

(b) Consider the group $(\mathbb{Z}/4, +, 0)$ where $\mathbb{Z}/4$ is the set $\{0, 1, 2, 3\}$ and $+$ is addition modulo 4 (for example $2 + 2 = 0$ and $2 + 3 = 1$). Write the multiplication table of $\mathbb{Z}/4$.

- **Remarks**
  - You are very much encouraged to work with other students. However, submit your work alone.
  - I will be happy to help you with the homeworks. Please visit me in my office hours.

Good Luck!