Math 541 Spring 2011
Preparation for the Final Test

<u>Remarks</u>

- Answer all the questions below.
- A definition is just a definition – there is no need to justify it. Just write it down.
- Unless it's a definition, answers should be written in the following format:
  - Write the statement and/or result. *Statement*:.........
  - Write the main points that will appear in your explanation or proof or computation. *Main points*:.........
  - Write the actual explanation or proof or computation. *Proof*:......... or *Computation*:.........

1. Lagrange's theorem.
   (a) (8) Let H be a subgroup of a group G. Define the index $[G : H]$ of H in G.
   (b) (16) State and prove Lagrange's theorem.
   (c) (9) Let G be a group of order p, where p is a prime. Show that G is cyclic.
2. Normal subgroups and quotient groups.
   (a) (8) Define the notion of a normal subgroup N of a group G.
   (b) (16) Answer the following:
      1. Define the set G/N, the operation $\circ$, and the identity $1_{G/N}$ such that $(G/N, \circ, 1_{G/N})$ is a group.
      2. Show that indeed $(G/N, \circ, 1_{G/N})$ is a group with your definitions, and in particular $\circ$ is well defined.
      3. Show that a group N is a normal subgroup of a group G if and only if there exists a group homomorphism $\varphi : G \to G'$ with kernel N.
   (c) (9) Answer the following:
      1. Quote the theorem on the relation between the groups $G/\ker(\varphi)$ and $\mathrm{Im}(\varphi)$ for a group homomorphism $\varphi : G \to G'$.
      2. Define the groups $O(3)$ and $SO(3)$. Show that $SO(3) \lhd O(3)$ and find a group G such that $O(3)/SO(3) \simeq G$.
3. Finite subgroups of $SO(3)$.

(a) (9) Find (i.e. draw as clear as possible) subsets $\widetilde{\mathcal{P}_n}$, $\mathcal{P}_k$, P, $C, D \subset \mathbb{R}^3$, with stabilizer subgroups $Stab_{SO(3)}(\widetilde{\mathcal{P}_n}) \simeq C_n$, where C_n is the cyclic group of order n, $Stab_{SO(3)}(\mathcal{P}_k) \simeq D_k$, where D_k is the dihedral group of order $2k$, $Stab_{SO(3)}(P) \simeq A_4$, where A_4 is the alternating group of even permutations of four letters, $Stab_{SO(3)}(C) \simeq S_4$, where S_4 is the group of permutations of four letters, and $Stab_{SO(3)}(D) \simeq A_5$, where A_5 is the group of even permutations of five letters.

(b) (16) Answer the following:

1. Let $T = Stab_{SO(3)}(\text{Pyramid})$. Find a natural isomorphism $r : T \to A_4$. Prove that indeed your r is an isomorphism.
2. Let $O = Stab_{SO(3)}(\text{Cube})$. Find a natural isomorphism $r : O \to S_4$. Prove that indeed your r is an isomorphism.
3. Let $I = Stab_{SO(3)}(\text{Dodecahedron})$. Find a natural isomorphism $r : I \to A_5$. Prove that indeed your r is an isomorphism (Remark: You can assume that I is simple).

(c) (9) Quote the theorem of Klein on the classification of the finite subgroups of $SO(3)$. Suppose $\Gamma \subset SO(3)$ is a subgroup of order 60 with an invariant vector for the action of Γ on $\mathbb{R}^3$, i.e., there exists $0 \neq v \in \mathbb{R}^3$ such that

$$Av = v, \quad \text{for every } A \in \Gamma.$$

Then $G \simeq$???

Good Luck!