Notes on April 11 Lecture

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Theorem 1. *If $N \lhd G$, (i.e. $\forall g \in G, gNg^{-1} = N$),*
then $\exists!$ group structure $\bullet$ on quotient G/N, which satisfies

$$\begin{aligned} \varphi: \quad & G \mapsto G/N \\ & g \mapsto gN \end{aligned}$$

is homomorphism of groups.

Claim. *The homomorphism $\varphi: \quad G \mapsto G'$ now induces an isomorphism:*

$$\begin{aligned} \bar{\varphi}: \quad & G/ker(\varphi) \simeq Im(\varphi) \\ & g \cdot ker(\varphi) \mapsto \varphi(g) \end{aligned}$$

Here is the proof of **Theorem 1**.

1. This map $\bullet$ exists and is unique (well-defined).

   (a) **Uniqueness (namely, $\bullet$ has to be defined of a very specific formula)**:

   Proof. Since we want φ to be a homomorphism under the group structure $\bullet$, we require

   $$\forall g, g' \in G, \quad \varphi(g) \bullet \varphi(g') = \varphi(gg')$$

   which is to say,

   $$\forall g, g' \in G, (gN) \bullet (g'N) = (gg')N$$

   So we can only define $\bullet$ in this way:

   $$\begin{aligned} \bullet: \quad & G/N \times G/N \mapsto G/N \\ & (gN, g'N) \mapsto (gg')N \end{aligned}$$

   □

(b) **Existence (the definition above is a well-defined map)** Since this definition involves representative element (i.e. we use g in the expressison of coset gN), to verify that it is well-defined, we should prove that the image of this map is independent of which representative element we choose. Namely, we want to verify that, if $aN = gN$, $a'N = g'N$, then $(aa')N = (gg')N$.

First, we need a lemma here.

Lemma. *If* $N \lhd G$*, then* $\forall g \in G, gN = Ng$.

Proof.

$$\begin{aligned} gN &= g(g^{-1}Ng) && (\text{by definition of normal group}, g^{-1}Ng = N) \\ &= (gg^{-1})Ng && (\text{Associativity}) \\ &= Ng && (\text{Inverse}) \end{aligned}$$

□

Now we are ready for the proof of the existence.

Proof. The proof is straightforward by the Lemma we proved above:

$$\begin{aligned} (aa')N &= a(a'N) && (\text{Associativity}) \\ &= a(g'N) && (a'N = g'N) \\ &= a(Ng') && (\textbf{Lemma}) \\ &= (aN)g' && (\text{Associativity}) \\ &= (gN)g' && (aN = gN) \\ &= g(Ng') && (\text{Associativity}) \\ &= g(g'N) && (\textbf{Lemma}) \\ &= (gg')N && (\text{Associativity}) \end{aligned}$$

□

2. $(G/N, \bullet, \text{something})$ is a group.

   (a) **Associativity**

$$\forall g_1, g_2, g_3 \in G, [(g_1N) \bullet (g_2N)] \bullet (g_3N) = (g_1N) \bullet [(g_2N) \bullet (g_3N)]$$

Proof.

$$\begin{aligned} [(g_1N) \bullet (g_2N)] \bullet (g_3N) &= (g_1g_2N) \bullet (g_3N) && (\text{definition of } \bullet) \\ &= [(g_1g_2)g_3]N && (\text{definition of } \bullet) \\ &= [g_1(g_2g_3)]N && (\text{Associativity}) \\ &= (g_1N) \bullet (g_2g_3N) && (\text{definition of } \bullet) \\ &= (g_1N) \bullet [(g_2N) \bullet (g_3N)] && (\text{definition of } \bullet) \end{aligned}$$

□

(b) **Identity**

$1_{G/N} \stackrel{\text{def}}{=} 1_G N = N$ (left coset of 1_G) is the identity of G/N.

Proof. Pick any $gN \in G/N$, we have

$$(gN) \bullet (1_G N) = (g \cdot 1_G)N = gN$$
$$(1_G N) \bullet (gN) = (1_G \cdot g)N = gN$$

□

(c) **Inverse**

$\forall gN \in G/N$, $g^{-1}N$ is the inverse of it.

Proof.

$$(gN) \bullet (g^{-1}N) = (gg^{-1})N = 1_G N = N$$
$$(g^{-1}N) \bullet (gN) = (g^{-1}g)N = 1_G N = N$$

Therefore $(gN)^{-1} = g^{-1}N$. □

3. **Note:** we can verify that φ is indeed a homomorphism since by definition

$$\varphi(gg') = (gg')N = (gN) \bullet (g'N) = \varphi(g) \bullet \varphi(g')$$

Now we can proceed to prove the claim, since we have already proved that $G/\ker(\varphi)$ is a group. Note that $\ker(\varphi)$ is normal, and $\forall g \in G, \bar{\varphi}(g \cdot \ker(\varphi)) = \{\varphi(g)\}$, which implies that $\bar{\varphi}$ is onto. Denote $N = \ker(\varphi)$. So we can find a natual defintion of $\bar{\varphi}$:

$$\begin{aligned} \bar{\varphi}: \quad & G/\ker(\varphi) \mapsto \text{Im}(\varphi) \\ & gN \mapsto \varphi(g) \end{aligned}$$

1. $\bar{\varphi}$ is homomorphism.

Proof. $\forall gN, g'N \in G/N$,

$$\begin{aligned} \bar{\varphi}[(gN) \bullet (g'N)] &= \bar{\varphi}(gg'N) && \text{(defnition of } \bullet) \\ &= \varphi(gg') && \text{(defnition of } \bar{\varphi}) \\ &= \varphi(g)\varphi(g') && (\varphi \text{ is homomorphism}) \\ &= \bar{\varphi}(gN)\bar{\varphi}(g'N) && \text{(defnition of } \bar{\varphi}) \end{aligned}$$

□

2. $\bar{\varphi}$ is a bijection.

The injection part is trivial to proof since $\ker(\bar{\varphi}) = N$, which is the identity of G/N.

Theorem 2. *If G is a group, N is a subgroup of G, then N is normal if and only if $\exists$ homomorphism $\varphi: \quad G \mapsto G'$, s.t. $N = ker(\varphi)$*

Proof. The proof is trivial.

$\Leftarrow$: Easy calculation.

$\Rightarrow$: Define map $\varphi : G \mapsto G/N$ as in **Theorem 1**.

$\square$

Quesion: How can we "see" G/N ?

Suppose $N \lhd G$, find a homomorphism $\varphi : G \mapsto G'$ with $\ker(\varphi) = N$, then $\mathrm{Im}(\varphi)$ is just isomorphism to G/N if we can explicityly compute $\mathrm{Im}(\varphi)$.

Example 1 $G = \mathrm{GL}_2(\mathbb{R})$, which is the set of all 2×2 invertible real matrices.

$N = \mathrm{SL}_2(\mathbb{R})$, set of all 2×2 invertible real matrices whose determinant$= 1$.

Since det is a homomorphism from G onto $\mathbb{R}^*$, and $\mathrm{Im}(\det)$ is exactly $\mathbb{R}^*$, so we can see that

$$\mathrm{GL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{R}) \simeq \mathbb{R}^*$$