Math 491 - Linear Algebra II, Fall 2015

Practice Final

May 11, 2015

Remarks

- Answer all the questions below. The best three (a), (b), and (c) will be counted towards your score.
- A definition is just a definition - there is no need to justify it. Just write it down.
- Unless it's a definition, answers should be written in the following format:
  - Write the main points that will appear in your proof of computation. *Main points:...*
  - Write the actual explanation or proof or computation. *Proof:...* or *Computation:...*

1. Primary Decomposition Theorem
   (a) (8) State precisely the Primary Decomposition Theorem.
   (b) (15) Let $T : V \to V$ be a linear transformation. Show that
   $$m_T(x) = (x - \lambda_1) \cdots (x - \lambda_s),$$
   for distinct $\lambda_i \in \mathbb{F}$ if and only if T is diagonalizable.
   (c) (10) Consider the transformation $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T_A(v) = Av$, where
   $$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 6 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$
   Compute $m_A(x)$ and use the criterion in (b) to decide if T_A is diagonalizable.

2. Jordan Decomposition
   (a) (8) Define the notion of a Jordan block, a Jordan matrix with one eigenvalue, and a Jordan matrix with multiple eigenvalues. State precisely the Jordan Theorem.

(b) (15) Write down all possible 4×4 Jordan matrices in $M_4(\mathbb{F})$.

(c) (10) In the following True and False, if you answer true, prove the statement, and if you answer false, provide a justified counterexample. Let $A, B \in M_4(\mathbb{C})$.

 (i) T / F – If $m_A = m_B$, then A and B are similar.

 (ii) T / F – If $m_A = m_B$ and $p_A = p_B$, then A and B are similar.

3. Similarity

 (a) (8) Define when two linear transformations $S, T : V \to V$ are similar, and when two matrices $A, B \in M_n(\mathbb{F})$ are similar.

 (b) (15) let $S, T : V \to V$ be two linear transformations of a vector space V over $\mathbb{F}$. Show that S and T are similar if and only if there exists bases $\mathcal{B}$ and $\mathcal{C}$ of V, such that the matrices $[S]_{\mathcal{B}}$ and $[T]_{\mathcal{C}}$ in $M_n(\mathbb{F})$ are similar.

 (c) (10) Let V be a vector space over $\mathbb{F} = \mathbb{C}$. Prove that two transformations $S, T : V \to V$ are similar if and only if they have the same Jordan Form (up to a permutation of the Jordan blocks).

4. Computing a Jordan Basis

 (a) (8) Let $T : V \to V$ be a linear transformation. Define the notion of a Jordan basis for V associated to T.

 (b) (15) Find a Jordan basis for $T_A : \mathbb{R}^4 \to \mathbb{R}^4$, where

$$A = \begin{pmatrix} 0 & -1 & 2 & -2 \\ 1 & 2 & -2 & 2 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

 (c) (10) Let A be the matrix given in 4(b). Compute A^{2015}.