Math 491 - Linear Algebra II, Fall 2015

Practice Final

May 3, 2016

Remarks

- Answer all the questions below. The best three (a), (b), and (c) will be counted towards your score.
- A definition is just a definition - there is no need to justify it. Just write it down.
- Unless it's a definition, answers should be written in the following format:
  - Write the main points that will appear in your proof of computation. *Main points:...*
  - Write the actual explanation or proof or computation. *Proof:...* or *Computation:...*

1. **Jordan Canonical Form**

   (a) (8) Define the following types of matrices:

   (i) A Jordan block $J_r(\lambda) \in M_r(\mathbb{C})$ associated to eigenvalue λ;

   (ii) A Jordan array $J(\lambda) \in M_k(\mathbb{C})$ associated to eigenvalue λ and the notion of index of an array;

   (iii) A Jordan matrix $J \in M_n(\mathbb{C})$.

   State precisely the Jordan canonical form theorem for an operator $T : V \to V$ where V is an n-dimensional vector space over $\mathbb{C}$.

   (b) (15) Assume that V is a vector space over $\mathbb{C}$ of dimension 4. Let $T : V \to V$ be a linear transformation.

   (i) Recall that there exists a Jordan basis $\mathcal{B}$ of V such that $[T]_{\mathcal{B}} = J$ where $J \in M_4(\mathbb{C})$ is a Jordan matrix. Show that, up to permutation of the Jordan blocks, J is the unique Jordan matrix for which a Jordan basis for T exists.

   (ii) Compute a Jordan basis for the transformation $T_A : \mathbb{R}^4 \to \mathbb{R}^4$ defined by $T_A(v) = Av$, where

$$A = \begin{pmatrix} 4 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 \\ -1 & 0 & 2 & 0 \\ 4 & 0 & 1 & 2 \end{pmatrix}.$$

(c) (10) Let $T, S : V \to V$ be two linear transformations on a finite dimensional vector space V over $\mathbb{F}$. Suppose that $m_T(x) = m_S(x)$, and

$$p_T(x) = p_S(x) = (x - \lambda_1)^{d_1} \cdots (x - \lambda_k)^{d_k}.$$

Suppose the $\lambda_i \in \mathbb{F}$ are distinct, and for all $1 \le i \le k$, $1 \le d_i \le 3$. Show that T and S are similar.

2. **Inner Product Spaces**

   (a) (8) Define the notion of an inner product space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.

   (b) (15) Define when two inner product spaces are isometric. Show that if $(V, \langle \cdot, \cdot \rangle)$ is an n-dimensional inner product space then it is isometric to $(\mathbb{F}^n, \langle \cdot, \cdot \rangle_{\text{st}})$.

   (c) (10) Let $V = \mathbb{R}_{\le 3}[x]$ be the vector space of real polynomials with degree at most 3. Given two polynomials, $f, g \in V$, recall that the pairing

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x)\, dx$$

   defines an inner product on V. Let $W = \text{Span}\{x^2, x + 1, x^3\}$. Use the Gram-Schmidt process to find an orthonormal basis for W.

3. **Adjoint Operator**

   (a) (8) Let T be a linear transformation on a finite dimensional inner product space $(V, \langle \cdot, \cdot \rangle)$. Define the adjoint operator $T^* : V \to V$.

   (b) (15) Let T be a linear transformation on a finite dimensional inner product space $(V, \langle \cdot, \cdot \rangle)$. Show that T^* is unique and prove its existence.

   (c) (10) Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. For $A \in M_n(\mathbb{F})$ define the matrix A^* and write a formula for it in terms of A. Let $(V, \langle \cdot, \cdot \rangle)$ be an n-dimensional inner product space over $\mathbb{F}$. Show that if $T : V \to V$ is a linear transformation and $\mathcal{B} \subset V$ is an orthonormal basis for V then

$$[T^*]_{\mathcal{B}} = [T]_{\mathcal{B}}^*.$$

4. **Spectral Theorem**

   Throughout this problem, assume that $(V, \langle \cdot, \cdot \rangle)$ is a finite dimensional inner product space over $\mathbb{C}$.

   (a) (8) Define the notion of a normal operator $T : V \to V$.

   (b) (15) State and then prove the spectral theorem for an operator T on V.

(c) (10) Let $V = \mathbb{C}(\mathbb{F}_5)$ be the vector space of complex valued functions on $\mathbb{F}_5$. Given two vectors $f, g \in V$, the pairing

$$\langle f, g \rangle = \sum_{x \in \mathbb{F}_5} f(x)\overline{g(x)},$$

defines an inner product on V. Let $\mathcal{F} : V \to V$ be the discrete Fourier transform, i.e. the linear transformation defined by

$$\mathcal{F}(f)(y) = \frac{1}{\sqrt{5}} \sum_{x \in \mathbb{F}_5} e^{\frac{2\pi i}{5} \cdot xy} f(x).$$

Show that $\mathcal{F}$ is unitary and diagonalizable.