Math 491 - Linear Algebra II, Fall 2016

Homework 9 - Inner Product Spaces and Gram-Schmidt

Quiz on 4/19/16

Remark: Answers should be written in the following format:
A) Result.
B) If possible, the name of the method you used.
C) The computation or proof.

1. **Orthogonal Sets are Linearly Indepedent.** Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$). Suppose that $S \subset V$ is an orthogonal set that does not contain the zero vector, i.e. for any distinct $u, v \in S$ we have $\langle u, v \rangle = 0$ (denoted $u \perp v$). Show that S is linearly independent over $\mathbb{F}$.

2. **A Couple of Identities.** Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$). Recall that we can define a norm on V by defining $||v|| = \sqrt{\langle v, v \rangle}$. Let $u, v \in V$. Show that
$$||u + v||^2 = ||u||^2 + ||v||^2 + \langle u, v \rangle + \langle v, u \rangle.$$
Use this identity to quickly deduce the following theorems. In each case, draw a picture that demonstrates the geometric meaning of each theorem.

   (i) (Parallelogram Identity)
$$||u + v||^2 + ||u - v||^2 = 2||u||^2 + 2||v||^2.$$

   (ii) (Pythagorean Theorem) Suppose $u \perp v$. Then
$$||u + v||^2 = ||u||^2 + ||v||^2.$$

3. **An Isomorphism to the Dual Space.** Let W be a finite dimensional vector space over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$). Denote by $\overline{W}$ the vector space with W as the set of vectors, the same addition of vectors, but where scalar multiplication is defined by:
$$a \star w = \bar{a} \cdot w$$
for any $a \in \mathbb{F}$ and $w \in W$, where $\bar{a}$ denotes the complex conjugate of a.

   Let V be a finite dimensional vector space over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$).

(a) Show that having an inner product on V induces an isomorphism from V to $\overline{V^*}$ where V^* is the dual space to V.

(b) Show that having an isomorphism from V to $\overline{V^*}$ defines an inner product on V.

4. **The Gram-Schmidt Algorithm.** Let $(V, \langle \cdot, \cdot \rangle)$ be a finite dimensional inner product space over $\mathbb{F}$ ($\mathbb{R}$ or $\mathbb{C}$) and let $W \subseteq V$ be a subspace of V. The Gram-Schmidt algorithm takes as input a basis for the subspace W and outputs an orthonormal basis for W.

   Before presenting the algorithm, we give a convenient definition. Let $u, v \in V$. Define the (orthogonal) projection of v onto u by

$$\mathrm{proj}_u(v) = \frac{\langle v, u \rangle}{\langle u, u \rangle} \cdot u.$$

   Gram-Schmidt Algorithm

Input: $\mathcal{B} = \{w_1, \ldots, w_k\}$ a basis for W.

(1) Set $u_1 = w_1$.
For $i = 2, \ldots, k$, set

$$u_i = w_i - \sum_{j=1}^{i-1} \mathrm{proj}_{u_j}(w_i).$$

(2) For $i = 1, \ldots, k$, replace u_i with $\frac{u_i}{||u_i||}$.

Output: $\mathcal{B}' := \{u_1, \ldots, u_k\}$.

(a) Show that $\mathcal{B}'$ is an orthonormal basis for W, i.e. that the algorithm returns a correct output. (Hint: Show that $\mathcal{B}'$ is an orthogonal set and apply problem 1.)

(b) In Matlab write an m-file that implements the above algorithm. Verify that your implementation is correct by checking its output on the example $V = \mathbb{R}^3$ with the usual inner product and

$$W = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} \right\}.$$

5. **Gram-Schmidt Examples.**

(a) Let $V = \mathbb{R}^4$ with inner product taken to be the dot product of two vectors. Use the Gram-Schmidt algorithm to compute an orthonormal basis of W where

$$W = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix} \right\}.$$

(b) Let $V = \mathbb{R}_{\leq 3}[x]$ be the vector space of real polynomials with degree at most 3. Recall that the function

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx$$

defines an inner product on V. Use the Gram-Schmidt algorithm to compute an orthonormal basis of V by beginning with the basis $\mathcal{B} = \{1, x, x^2, x^3\}$.