Math 491 - Linear Algebra II, Fall 2015

Homework 7 - More on Diagonalization and Related Topics

Quiz on 4/9/15

Remark: Answers should be written in the following format:
A) Result.
B) If possible, the name of the method you used.
C) The computation or proof.

1. **Diagonalizable Operators on Invariant Subspaces.** Let $T : V \to V$ be a diagonalizable operator on a finite dimensional vector space V over a field $\mathbb{F}$. Suppose $W \subset V$ is a T-invariant subspace. Show that $T|_W$ is diagonalizable by considering the minimal polynomial $m_{T|_W}$.

   Hint: Use the fact that T is diagonalizable if and only if its minimal polynomial is the product of distinct monic linear polynomials.

2. **Simultaneous Diagonalizability.** Let $S, T : V \to V$ be linear transformations. We say that S, T are <u>simultaneously diagonalizable</u> if there exists a direct sum decomposition $V = \oplus_{i=1}^{k} V_i$, and scalars $\lambda_i, \mu_i \in \mathbb{F}$, such that $T|_{V_i} = \lambda_i Id_{V_i}$, $S|_{V_i} = \mu_i Id_{V_i}$ for $i = 1, ..., k$.

   (a) Assume that S, T are diagonalizable. Show that $ST = TS$ if and only if S, T are simultaneously diagonalizable. Recall, that you showed one direction of this in a previous HW.

   (b) Let V be a finite dimensional vector space over a field $\mathbb{F}$. Denote by $L(V)$ the set of linear transformations from V to itself.

      (i) Show that $L(V)$ is an algebra over $\mathbb{F}$ in a natural way. That is, it has a natural addition and multiplication.

      (ii) Let $\mathcal{C} \subset L(V)$ be a subalgebra, i.e. closed under multiplication and addition, consisting of diagonalizable operators. Show that all elements of $\mathcal{C}$ are simultanteously diagonalizable if and only if $\mathcal{C}$ is commutative.

   Here simultaneously diagonalizable means there exists a decomposition $V = \oplus_{i=1}^{k} V_i$ such that $T|_{V_i} = \lambda_{T_i} \cdot Id_{V_i}$ for any $T \in \mathcal{C}$. Recall, an algebra $\mathcal{C}$ is commutative if for any $C_1, C_2 \in \mathcal{C}$ we have $C_1 C_2 = C_2 C_1$.

3. **Nilpotent Operators.** Let $T : V \to V$ be a linear transformation on a finite dimensional vector space V over $\mathbb{F}$. We say that T is <u>nilpotent</u> if there exists a flag of subspaces of V,
$$\{0_V\} = V_0 \subset V_1 \subset \cdots \subset V_k = V,$$
such that $T(V_i) \subset V_{i-1}$ for all $i = 1, ..., k$.

   (a) Show the following operators are nilpotent by constructing a flag as above.

      (i) The derivative operator D on $\mathbb{F}_{\leq 3}[x]$, the space of polynomials over $\mathbb{F}$ of degree at most 3.

      (ii) The operator $T_A : \mathbb{R}^3 \to \mathbb{R}^3$, defined by multiplication by
$$A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}.$$

   (b) Show T is nilpotent if and only if $T^k = 0$ for some integer $k > 0$.

   (c) Show T is nilpotent if and only if there is a basis $\mathcal{B}$ of V such that $[T]_{\mathcal{B}}$ is a strictly upper-triangular matrix, i.e. all entries on and below the diagonal are 0.

   (d) Let V be a vector space over $\mathbb{F}$. Consider a flag of the form:
$$\mathcal{F} : \ \{0_V\} = V_0 \subset V_1 \subset \cdots \subset V_k = V.$$
   Denote by $N_{\mathcal{F}}$ the set of all linear transformations $T : V \to V$, such that $T(V_i) \subset V_{i-1}$ for $i = 1, ..., k$. Show $N_{\mathcal{F}}$ is a subalgebra of $L(V)$, i.e. it is closed under addition and composition.

   (e) Let $\mathcal{N} \subset L(V)$ be a maximal collection of commuting nilpotent operators. Show that $\mathcal{N}$ is a subalgebra.

4. **Spectral Decomposition.** Let $T : V \to V$ be a diagonalizable operator on a finite dimensional vector space V over $\mathbb{F}$. Then
$$V = \bigoplus_{\lambda \in \mathrm{Spec}(T)} V_\lambda,$$
where $V_\lambda = \ker(T - \lambda Id)$. Denote by P_λ the projector on V defined by $P_\lambda|_{V_\lambda} = Id|_{V_\lambda}$, and $P_\lambda|_{V_\mu} = 0$ for every $\mu \neq \lambda$.

   (a) Show
$$P_\lambda = \frac{1}{\prod_{\mu \neq \lambda}(\lambda - \mu)} \prod_{\mu \neq \lambda} (T - \mu \cdot Id),$$
   where the products range over $\mu \in \mathrm{Spec}(T)$ and $\mu \neq \lambda$.

(b) Show
$$T = \sum_{\lambda \in \mathrm{Spec}(T)} \lambda \cdot P_\lambda.$$

This decomposition is sometimes called the <u>spectral decomposition</u> of T.

Remark
The grader and the Lecturer will be happy to help you with the homework. Please visit office hours.

Good luck!