Math 491 - Linear Algebra II, Fall 2016

Homework 5 - Quotient Spaces and Cayley–Hamilton Theorem

Quiz on 3/8/16

Remark: Answers should be written in the following format:
A) Result.
B) If possible, the name of the method you used.
C) The computation or proof.

1. **Quotient space.** Let V be a vector space over a field $\mathbb{F}$ and $W \subset V$ subspace. For every $v \in V$ denote by $v + W$ the set

$$v + W = \{v + w \mid w \in W\},$$

and call it the coset of v with respect to W. Denote by V/W the collection of all cosets of vectors of V with respect to W, and call it the quotient of V by W.

   (a) Define the operation $\boldsymbol{+}$ on V/W by setting,

$$(v + W)\boldsymbol{+}(u + W) = (v + u) + W.$$

   Show that $\boldsymbol{+}$ is well-defined. That is, if $v, v' \in V$ and $u, u' \in V$ are such that $v + W = v' + W$ and $u + W = u' + W$, show

$$(v + W)\boldsymbol{+}(u + W) = (v' + W)\boldsymbol{+}(u' + W).$$

   Hint: It may help to first show that for vectors $v, v' \in V$, we have $v + W = v' + W$ if and only if $v - v' \in W$.

   (b) Define the operation $\cdot$ on V/W by setting,

$$\alpha \cdot (v + W) = \alpha \cdot v + W.$$

   Show that $\cdot$ is well-defined. That is, if $v, v' \in V/W$ and $\alpha \in \mathbb{F}$ such that $v + W = v' + W$, show that

$$\alpha \cdot (v + W) = \alpha \cdot (v' + W).$$

(c) Define $0_{V/W} \in V/W$ by $0_{V/W} = 0_V + W$. Show that $0_{V/W}$ acts as a zero vector in V/W. That is, for any $v + W \in V/W$, we have

$$(v+W) + 0_{V/W} = 0_{V/W} + (v+W) = v + W.$$

(d) It can be shown that the quadruple $(V/W, +, \cdot, 0_{V/W})$ is a vector space over $\mathbb{F}$. Assume now that V is finite dimensional. Show that

$$\dim(V/W) = \dim(V) - \dim(W).$$

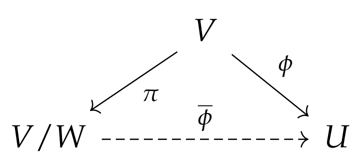
Hint: If $\dim(W) = k$, you may wish to extend a basis $\{w_1, ..., w_k\}$ of W to a basis of V.

2. **A Universal Property.** Let V be a vector space over $\mathbb{F}$ and $W \subset V$ a subspace. Consider the canonical projection

$$\pi : V \to V/W,$$
$$v \mapsto v + W.$$

(a) Show that $\ker(\pi) = W$.

(b) We will show that the linear transformation π satisfies the following property: For every vector space U and linear transformation $\phi : V \to U$, such that $W \subset \ker(\phi)$, there exists a unique linear transformation $\overline{\phi} : V/W \to U$ such that the following diagram commutes:

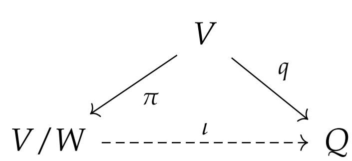


i.e., $\overline{\phi} \circ \pi = \phi$. To do this,

(i) Show that there exists at most one such linear transformation $\overline{\phi}$. What must be its formula?

(ii) Construct such a $\overline{\phi}$.

(c) **The Universal Property.** Let Q be a vector space. Suppose we have a linear transformation $q : V \to Q$ that satisfies the same property as π above. Show that there is a unique isomorphism $\iota : V/W \xrightarrow{\sim} Q$ such that $\iota \circ \pi = q$,



3. **Induced Linear Transformations.** Let $T : V \to V$ be a linear transformation on a vector space V and let $W \subset V$ be a subspace such that $T(W) \subset W$, that is, W is T-invariant. Define a map

$$\begin{aligned} \overline{T} : V/W &\to V/W, \\ v + W &\mapsto T(v) + W. \end{aligned}$$

Show that $\overline{T}$ is well-defined, i.e. independent of representatives. We define a representative of a coset $v + W$ as any vector $v' \in V$ such that $v + W = v' + W$.

4. **Cayley–Hamilton Theorem** This exercise will guide you through a proof of the following theorem:

Theorem (Cayley–Hamilton) Let $T : V \to V$ be a linear transformation on a finite dimensional vector space V over a field $\mathbb{F}$. Then $p_T(T) = 0$.

In class, we will show that the general statement follows from the case when $\mathbb{F}$ is algebraically closed, and this is what we assume from now on. Prove the theorem using the following steps:

(a) Show that the theorem is true for $T_U : \mathbb{F}^n \to \mathbb{F}^n$ where

$$U = \begin{pmatrix} \lambda_1 & * & * & * \\ & \lambda_2 & \ddots & * \\ & & \ddots & * \\ & 0 & & \lambda_n \end{pmatrix}$$

Hint: Apply $p_{T_U}(U)$ on the standard basis vectors $e_n, ..., e_1$.

(b) Show that the theorem is true if there exists a "flag" in V invariant under T, i.e., a sequence of subspaces

$$\{0_V\} \subset V_1 \subset V_2 \subset \cdots \subset V_n = V,$$

such that $\dim(V_i) = i$, and $T(V_i) \subset V_i$ for all $i = 1, ..., n$. Hint: Take a basis $\mathcal{B} = \{v_1, ..., v_n\}$ of V with $\mathcal{B}_i = \{v_1, ..., v_i\}$ a basis of V_i for each i, and consider $[T]_{\mathcal{B}} = U$.

(c) Show by induction on $\dim(V) = n$ that such a flag exists for any linear transformation $T : V \to V$. The case $\dim(V) = 1$ is trivial. Now assume the theorem holds for any linear transformation on a vectors space of dimension less than n. Complete the induction step in the following way:

(i) Show that $P_T(X)$ has a root λ_1 and an associated eigenvector v_1. Construct the subspace $V_1 = \mathrm{Span}(\{v_1\})$ and note that it is T-invariant.

(ii) Consider the induced linear transformation $\overline{T} : V/V_1 \to V/V_1$ and note $\dim(V/V_1) = n-1$. By the induction assumption, there is a flag of $\overline{T}$-invariant subspaces

$$\{0_{V/V_1}\} \subset \overline{V_2} \subset \overline{V_3} \subset \cdots \subset \overline{V_n} = V/V_1.$$

Recall, there is a canonical projection $\pi : V \to V/V_1$ and for any subset $S \subset V/V_1$, we have $\pi^{-1}(S) = \{v \in V \mid \pi(v) \in S\}$. Show that the sequence

$$\{0_V\} \subset V_1 \subset \pi^{-1}(\overline{V_2}) \subset \cdots \pi^{-1}(\overline{V_n}) = V,$$

is a T-invariant flag in V.