Math 491 - Linear Algebra II, Fall 2015

Homework 4 - Factorization, Eigenvalues, and Diagonalization

Quiz on 3/1/16

Remark: Answers should be written in the following format:
A) Result.
B) If possible, the name of the method you used.
C) The computation or proof.

1. Let

$$\begin{aligned} f(X) &= X^3 - 6X^2 + 11X - 6 \\ g(X) &= X^3 - 4X^2 + 5X - 2 \\ h(X) &= X^3 - 5X^2 + 8X - 4 \end{aligned}$$

   be three real polynomials. Compute the greatest common divisor of f, g, h.

2. Let $T : \mathbb{F}_2^4 \to \mathbb{F}_2^4$ be the linear transformation defined by

$$(x_1, x_2, x_3, x_4) \mapsto (x_1 + x_3, -x_1 - x_3, x_2 + x_4, x_2 - x_4).$$

   Find all eigenvalues of T and for each eigenvalue, compute the corresponding eigenspace.

3. For each of the possible values of the matrix $A \in M_3(\mathbb{Q})$ below,

$$\begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

   find their eigenvalues (in $\mathbb{Q}$) and compute the corresponding eigenspaces. Then decide if there exists an invertible 3×3 matrix C such that $C^{-1}AC$ is diagonal. If such a matrix C exists, find C and compute $C^{-1}AC$.

4. (a) Let $\mathbb{F}$ be an algebraically closed field. Let V be a vector space over $\mathbb{F}$ and $T : V \to V$ a linear transformation. Show that T has at least one eigenvalue.
   (b) Find an example of a non-algebraically closed field $\mathbb{F}$, a vector space V over $\mathbb{F}$, and a linear transformation $T : V \to V$ such that T has no eigenvalues.

5. Let $P : V \to V$ be a linear transformation on a vector space V over a field $\mathbb{F}$ such that $P^2 = P$. Show that P is diagonalizable.

6. Let $T : V \to V$ be a linear transformation, where V is a vector space of dimension n over a field $\mathbb{F}$. A subspace $W \subset V$ is called T-invariant if $T(W) \subset W$. Suppose that in addition we have another operator $S : V \to V$ and that S and T commute, i.e. $ST = TS$. Show that

   (a) If $W = W_\mu$ is an eigenspace for S associated with the eigenvalue $\mu \in \mathbb{F}$, then W is T-invariant.

   (b) If S, T are each diagonalizable, then they are simultaneously diagonalizable, i.e., there exist scalars $\mu_i, \lambda_i \in \mathbb{F}$, for $i = 1, ..., k$, and a direct sum decomposition

   $$V = \bigoplus_{i=1}^{k} V_i,$$

   such that $S|_{V_i} = \mu_i Id_{V_i}$ and $T|_{V_i} = \lambda_i Id_{V_i}$ for every $i = 1, ..., k$.

7. Let $T : V \to V$ be a linear transformation on a finite dimensional vector space V over a field $\mathbb{F}$. Assume there exists $\mu_1, \ldots, \mu_k \in \mathbb{F}$ and T-invariant subspaces $V_1, \ldots, V_k$ of V such that $V = V_1 \oplus \cdots \oplus V_k$, and for each $i = 1, \ldots, k$ we have $T|_{V_i} = \mu_i \cdot Id_{V_i}$. Show that

   $$V = \bigoplus_{\lambda \in \mathrm{Spec}(T)} V_\lambda,$$

   where V_λ is the eigenspace associated to λ, i.e., $V_\lambda = \{v \in V \mid T(v) = \lambda v\}$.

Remark

The grader and the Lecturer will be happy to help you with the homework. Please visit office hours.

Good luck!