Math 491 - Linear Algebra II, Fall 2015

Homework 4 - Eigenvalues and Diagonalization

Quiz on 3/5/15

Remark: Answers should be written in the following format:
A) Result.
B) If possible, the name of the method you used.
C) The computation or proof.

1. Let $T : \mathbb{F}_2^4 \to \mathbb{F}_2^4$ be the linear transformation defined by

$$(x_1, x_2, x_3, x_4) \mapsto (x_1 + x_3, -x_1 - x_3, x_2 + x_4, x_2 - x_4).$$

   Find all eigenvalues of T and for each eigenvalue, compute the corresponding eigenspace.

2. For each of the possible values of the matrix $A \in M_3(\mathbb{Q})$ below,

$$\begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & -1 \\ -1 & 1 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

   find their eigenvalues (in $\mathbb{Q}$) and compute the corresponding eigenspaces. Then decide if there exists an invertible 3×3 matrix C such that $C^{-1}AC$ is diagonal. If such a matrix C exists, find C and compute $C^{-1}AC$.

3. (a) Let $\mathbb{F}$ be an algebraically closed field. Let V be a vector space over $\mathbb{F}$ and $T : V \to V$ a linear transformation. Show that T has at least one eigenvalue.

   (b) Find an example of a non-algebraically closed field $\mathbb{F}$, a vector space V over $\mathbb{F}$, and a linear transformation $T : V \to V$ such that T has no eigenvalues.

4. Let $P : V \to V$ be a linear transformation on a vector space V over a field $\mathbb{F}$ such that $P^2 = P$. Show that P is diagonalizable.

5. Let $T : V \to V$ be a linear transformation, where V is a vector space of dimension n over a field $\mathbb{F}$. A subspace $W \subset V$ is called <u>T-invariant</u> if $T(W) \subset W$. Suppose that in addition we have another operator $S : V \to V$ and that S and T commute, i.e. $ST = TS$. Show that

   (a) If $W = W_\mu$ is an eigenspace for S associated with the eigenvalue $\mu \in \mathbb{F}$, then W is T-invariant.

   (b) If S, T are each diagonalizable, then they are <u>simultaneously</u> diagonalizable, i.e., there exist scalars $\mu_i, \lambda_i \in \mathbb{F}$, for $i = 1, ..., k$, and a direct sum decomposition

$$V = \bigoplus_{i=1}^{k} V_i,$$

   such that $S|_{V_i} = \mu_i Id_{V_i}$ and $T|_{V_i} = \lambda_i Id_{V_i}$ for every $i = 1, ..., k$.

Remark
The grader and the Lecturer will be happy to help you with the homework. Please visit office hours.

Good luck!