Math 491 - Linear Algebra II, Fall 2016

Homework 3 - Factorization and Finding Eigenvalues

Quiz on 2/23/16

Remark: Answers should be written in the following format:
A) Result.
B) If possible, the name of the method you used.
C) The computation.

Theoretical Exercises

1. **The ring of integers is a PID**. Prove that the ring of integers $\mathbb{Z}$ is a principal ideal domain. That is, show that every ideal I of $\mathbb{Z}$ is generated by a single element, i.e. $I = (n)$ for some $n \in \mathbb{Z}$.

2. **Factoring real polynomials in $\mathbb{C}[X]$.** Let $f(X) = X^2 + bX + c \in \mathbb{R}[X]$. Factor $f(X)$ into linear polynomials in $\mathbb{C}[X]$. Hint: Try the quadratic formula.

3. **Factoring in $\mathbb{R}[x]$.** Recall, that an element $r \in R$ is irreducible if for any $a, b \in R$ such that $r = ab$ then either $a \in R^\times$ or $b \in R^\times$. Factor the following polynomials from $\mathbb{R}[X]$ into irreducibles.

   (a) $f(X) = X^4 + 1$

   (b) $f(X) = X^6 - 1$

4. **Ideals in $M_n(\mathbb{F})$.** Let $\mathbb{F}$ be a field and let $R = M_n(\mathbb{F})$ be the ring of $n \times n$ matrices over $\mathbb{F}$. Show that there are no ideals (two-sided) in R other than $\{0\}$ and R. Give an example of a non-trivial left ideal.

5. **Irreducibles need not be primes.** Recall that in an integral domain R, a nonzero non-unit $q \in R$ is <u>prime</u> if whenever $q \mid ab$ then either $q \mid a$ or $q \mid b$. A nonzero non-unit $q \in R$ is <u>irreducible</u> if whenever $q = ab$ then either a is a unit or b is a unit. Consider the subring S of $\mathbb{C}$,

$$S = \{a + b\sqrt{-3} \mid a, b \in \mathbb{Z}\}.$$

   Show that in this integral domain, $2 \in S$ is irreducible but not prime.

6. **Eigenvalues over $\mathbb{R}$ and $\mathbb{C}$.** For each of the following linear transformations, find all eigenvalues. For each eigenvalue, find the corresponding eigenspace. In each case, do the problem first with $\mathbb{F} = \mathbb{R}$ and then again with $\mathbb{F} = \mathbb{C}$.

   (a) $T : \mathbb{F}^3 \to \mathbb{F}^3$,
   $$T(x_1, x_2, x_3) = (x_1 + x_2, x_2 + x_3, x_1 + x_3).$$

   (b) $T : \mathbb{F}^2 \to \mathbb{F}^2$,
   $$T(x_1, x_2) = (x_1 + x_2, x_1 - x_2).$$

   (c) $T : \mathbb{F}^4 \to \mathbb{F}^4$,
   $$T(x_1, x_2, x_3, x_4) = (x_2, 2x_3, 3x_4, 0).$$

Numerical Exercises

7. **Factoring polynomials with software**. In this exercise, you will use the software MATLAB to factor the polynomial $f(X) = X^3 - 2$.

   (a) To have MATLAB factor $f(X)$ type the following commands.

   ```
   syms x;
   factor(x^3-2)
   ```

   MATLAB seems to fail immediately. This is because MATLAB factors in $\mathbb{Q}[X]$ by default. Show that $f(X)$ is irreducible over $\mathbb{Q}[X]$.

   (b) MATLAB can factor over both $\mathbb{R}$ and $\mathbb{C}$ as well. To factor $f(X)$ in MATLAB, type the following commands

   ```
   syms x;
   factor(x^3-2, 'Factormode', 'real')
   factor(x^3-2, 'Factormode', 'complex')
   ```

   Notice that over the reals $f(X)$ factors as the product of a linear and quadratic polynomial, but splits into three linear factors over the complex numbers. Neither of these presentations is very satisfactory as MATLAB is just approximating cube roots with decimals. Provide an exact factorization of $f(X)$ over $\mathbb{R}$.

8. **Diagonalization with software**. In this exercise, you will use the software MATLAB to find the eigenvectors of a given transformation. Let $V = \mathbb{R}_{\leq 2}[X]$ be the vector space of polynomials of degree at most 2. Let $T : V \to V$ be the linear transformation defined by $T(p(X)) = (Xp(X))'$.

   (a) Let $\mathcal{B} = \{X + 1, X - 1, X^2 + X - 1\}$. Compute $[T]_{\mathcal{B}}$ and store this matrix in MATLAB as T_B.

   (b) To diagonalize T_B with MATLAB, type the following commands.

```
[C,D] = eig(T_B);
```

Use MATLAB to verify that $CDC^{-1} = [T]_{\mathcal{B}}$. Find a basis consisting of eigenvectors of T by using the matrix C and the basis $\mathcal{B}$.

Remark
The grader and the Lecturer will be happy to help you with the homework. Please visit office hours.

Good luck!