Math 491 - Linear Algebra II, Fall 2015

Homework 3 - Factorization and Finding Eigenvalues

Quiz on 2/26/15

Remark: Answers should be written in the following format:
A) Result.
B) If possible, the name of the method you used.
C) The computation.

1. **Factoring real polynomials in $\mathbb{C}[X]$.** Let $f(X) = X^2 + bX + c \in \mathbb{R}[X]$. Factor $f(X)$ into linear polynomials in $\mathbb{C}[X]$. Hint: Try the quadratic formula.

2. **Factoring in $\mathbb{R}[x]$.** Recall, that an element $r \in R$ is irreducible if for any $a, b \in R$ such that $r = ab$ then either $a \in R^\times$ or $b \in R^\times$. Factor the following polynomials from $\mathbb{R}[X]$ into irreducibles.

   (a) $f(X) = X^4 + 1$
   (b) $f(X) = X^6 - 1$

3. **Ideals in $M_n(\mathbb{F})$.** Let $\mathbb{F}$ be a field and let $R = M_n(\mathbb{F})$ be the ring of $n \times n$ matrices over $\mathbb{F}$. Show that that are no ideals (two-sided) in R other than $\{0\}$ and R.

4. **Irreducibles need not be primes.** Recall that in an integral domain R, a nonzero non-unit $q \in R$ is prime if whenever $q \mid ab$ then either $q \mid a$ or $q \mid b$. Consider the subring S of $\mathbb{C}$,
$$S = \{a + b\sqrt{-3} \mid a, b \in \mathbb{Z}\}.$$
Show that in this integral domain, $2 \in S$ is irreducible but not prime.

5. **Eigenvalues over $\mathbb{R}$ and $\mathbb{C}$.** For each of the following linear transformations, find all eigenvalues. For each eigenvalue, find the corresponding case. In each case, do the problem first with $\mathbb{F} = \mathbb{R}$ and then again with $\mathbb{F} = \mathbb{C}$.

   (a) $T : \mathbb{F}^3 \to \mathbb{F}^3$,
$$T(x_1, x_2, x_3) = (x_1 + x_2, x_2 + x_3, x_1 + x_3).$$
   (b) $T : \mathbb{F}^2 \to \mathbb{F}^2$,
$$T(x_1, x_2) = (x_1 + x_2, x_1 - x_2).$$

(c) $T : \mathbb{F}^4 \to \mathbb{F}^4$,

$$T(x_1, x_2, x_3, x_4) = (x_2, 2x_3, 3x_4, 0).$$

Remark

The grader and the Lecturer will be happy to help you with the homework. Please visit office hours.

Good luck!