Math 491 - Linear Algebra II, Spring 2016

Homework 2 - Rings

Quiz on 2/9/16 with Review Test

Remark: Answers should be written in the following format:
A) Result.
B) If possible, the name of the method you used.
C) The computation.

1. **Degree of a polynomial.** Let $\mathbb{F}$ be a field and $\mathbb{F}[X]$ the ring of polynomials with coefficients in $\mathbb{F}$.
   (a) Show that $\dim \mathbb{F}[X] = \infty$.
   (b) Now let R be a ring and $R[X]$ the ring of polynomials with coefficients in R. Recall that the degree $\deg(f)$ of a polynomial $f \in R[X]$ is defined to be $\deg(f) = d$ if $f = a_d X^d + ... + a_1 X + a_0$ and $a_d \neq 0$, and $\deg(f) = -\infty$ if $f = 0$. Show that for every $f, g \in R[X]$ we have
      (i) $\deg(fg) \leq \deg(f) + \deg(g)$,
      (ii) $\deg(f+g) \leq \max\{\deg(f), \deg(g)\}$.

      Moreover, show that if R is an integral domain than the inequality in (i) is actually an equality.

2. **Kernel of a homomorphism.** Let $\varphi : R \to S$ be homomorphism of rings. Define the <u>kernel</u> of φ, denoted $\ker(\varphi)$, by $\ker(\varphi) = \{a \in R \text{ such that } \varphi(a) = 0\}$. Show that φ is one-to-one if and only if $\ker(\varphi) = \{0\}$.

3. **Uniqueness of inverses.** Show that if R is a ring with unit and $a \in R$ is invertible, i.e., there exists $b \in R$ such that $ab = ba = 1_R$, then it has a unique inverse.

4. **Subrings from ring homomorphisms.** Let $\varphi : R \to S$ be a homomorphism of rings. We define the image of φ, denoted $\text{Im}(\varphi)$, by $\text{Im}(\varphi) = \{\varphi(r) \mid r \in R\}$. Show that $\text{Im}(\varphi)$ is a subring of S and that $\ker(\varphi)$ is a subring of R.

5. **Kernel is an ideal.** Let R be a ring. A subset $I \subset R$ is called an <u>ideal</u> of R if I satisfies
   (i) $0_R \in I$

(ii) for all $a, b \in I$, we have $a + b \in I$

(iii) for all $a \in I$, we have $-a \in I$

(iv) for all $a \in I$ and $r \in R$ we have $r \cdot a \in I$

(v) for all $a \in I$ and $r \in R$ we have $a \cdot r \in I$

Now, let $\varphi : R \to S$ be a homomorphism of rings. Show that $\ker(\varphi) \subset R$ is an ideal of R.

6. **Inverse of a Homomorphism.** Let $\varphi : R \to S$ be a homomorphism of rings. We say that φ is invertible if there exists a ring homomorphism $\sigma : S \to R$ such that $\sigma \circ \varphi = id_R$ and $\varphi \circ \sigma = id_S$. Show that the following are equivalent:

   (i) φ is invertible

   (ii) φ is one-to-one and onto

   Hint for (ii) implies (i): Note that $\varphi^{-1} : S \to R$ exists. Show that φ^{-1} is a homomorphism of rings.

7. **Division Algorithm.** Let $\mathbb{F}$ be a field. Recall from class, if $f, g \in \mathbb{F}[X]$ then there exists unique polynomials $q, r \in \mathbb{F}[X]$ with $0 \leq \deg(r) < \deg(g)$ such that

$$f = qg + r.$$

   Find q, r in the following cases.

   (i) Let $\mathbb{F} = \mathbb{F}_7$, the field with 7 elements, and take $g = X^3 + X + 1$, $f = X^5 + 2X^4 - 3X^3 + X^2 - 1$.

   (ii) Let $\mathbb{F} = \mathbb{F}_2$, the field with 2 elements, and take $g = X + 1$, $f = X^3 + X$.

8. **Ring homomorphisms and Inverses.** Let R, S be rings with unit. Let $\varphi : R \to S$ be a homomorphism of rings with unit, that is $\varphi(1_R) = 1_S$. Recall that $R^\times = \{a \in R \mid a \text{ is invertible}\}$. Show that for every $a \in R^\times$, we have $\varphi(a) \in S^\times$. That is, φ maps invertible elements to invertible elements.

9. **Ideals in Polynomial Rings.** Let $\mathbb{F}$ be a field and $R = \mathbb{F}[X]$. Decide whether or not the following subsets of R are ideals.

   (a) $\{f \in R \mid \deg(f) < m\} \cup \{0\}$.

   (b) $\{f \in R \mid \deg(f) > m\} \cup \{0\}$.

   (c) $\{f = \sum a_i X^i \mid a_0 = 0\}$.

   (d) $\{f \in R \mid f(1) = 0\}$.

   (e) $\{f \in R \mid f(0) = 0\}$.

   (f) $\{f \in R \mid f(0) = f(1)\}$.

(g) $\{f \in R \mid f(0) = f(1) = 0\}$.

Remark
The grader and the Lecturer will be happy to help you with the homework. Please visit office hours.

Good luck!