Math 491 - Linear Algebra II, Fall 2015

Homework 2 - Rings and Things

Quiz on 2/19/15

Remark: Answers should be written in the following format:
A) Result.
B) If possible, the name of the method you used.
C) The computation.

1. **Subrings from ring homomorphisms.** Let $\phi : R \to S$ be a homomorphism of rings. We define the image of ϕ, denoted $\text{Im}(\phi)$, by $\text{Im}(\phi) = \{\phi(r) \mid r \in R\}$. Show that $\text{Im}(\phi)$ is a subring of S and that $\ker(\phi)$ is a subring of R.

2. **Kernel is an ideal.** Let R be a ring. A subset $I \subset R$ is called an <u>ideal</u> of R if I satisfies

   (i) $0_R \in I$

   (ii) for all $a, b \in I$, we have $a + b \in I$

   (iii) for all $a \in I$, we have $-a \in I$

   (iv) for all $a \in I$ and $r \in R$ we have $r \cdot a \in I$

   (v) for all $a \in I$ and $r \in R$ we have $a \cdot r \in I$

   Now, let $\phi : R \to S$ be a homomorphism of rings. Show that $\ker(\phi) \subset R$ is an ideal of R.

3. **Inverse of a Homomorphism.** Let $\phi : R \to S$ be a homomorphism of rings. We say that ϕ is <u>invertible</u> if there exists a ring homomorphism $\sigma : S \to R$ such that $\sigma \circ \phi = id_R$ and $\phi \circ \sigma = id_S$. Show that the following are equivalent:

   (i) ϕ is invertible

   (ii) ϕ is one-to-one and onto

   Hint for (ii) implies (i): Note that $\phi^{-1} : S \to R$ exists. Show that ϕ^{-1} is a homomorphism of rings.

4. **Division Algorithm.** Let $\mathbb{F}$ be a field. Recall, if $f, g \in \mathbb{F}[X]$ then there exists unique polynomials $q, r \in \mathbb{F}[X]$ with $0 \leq \deg(r) < \deg(g)$ such that

$$f = qg + r.$$

Find q, r in the following cases.

(i) Let $\mathbb{F} = \mathbb{F}_7$, the field with 7 elements, and take $g = X^3 + X + 1$, $f = X^5 + 2X^4 - 3X^3 + X^2 - 1$.

(ii) Let $\mathbb{F} = \mathbb{F}_2$, the field with 2 elements, and take $g = X + 1$, $f = X^3 + X$.

5. **Unitary Ring homomorphisms and Inverses.** Let R, S be rings with unit. Let $\phi : R \to S$ be a homomorphism of rings with unit, that is $\phi(1_R) = 1_S$. Recall that $R^\times = \{a \in R \mid a \text{ is invertible}\}$. Show that for every $a \in R^\times$, we have $\phi(a) \in S^\times$. That is, ϕ maps invertible elements to invertible elements.

6. **Ideals in Polynomial Rings.** Let $\mathbb{F}$ be a field and $R = \mathbb{F}[X]$. Decide whether or not the following subsets of R are ideals.

(a) $\{f \in R \mid \deg(f) < m\} \cup \{0\}$

(b) $\{f \in R \mid \deg(f) > m\} \cup \{0\}$

(c) $\{f = \sum a_i X^i \mid a_0 = 0\}$

(d) $\{f \in R \mid f(1) = 0\}$

(e) $\{f \in R \mid f(0) = 0\}$

(f) $\{f \in R \mid f(0) = f(1)\}$

(g) $\{f \in R \mid f(0) = f(1) = 0\}$

Remark

The grader and the Lecturer will be happy to help you with the homework. Please visit office hours.

Good luck!