Math 491 - Linear Algebra II, Fall 2016

Homework 10 - Orthogonal Projection and Unitary Operators

Quiz on 4/26/16

Remark: Answers should be written in the following format:
A) Result.
B) If possible, the name of the method you used.
C) The computation or proof.

1. We know (exercise 3 below) that a complex $n \times n$ matrix A is unitary if and only if $A^*A = I$. Here A^* denotes the conjugate transpose of A. Show that the following are equivalent.

   (i) A is unitary.

   (ii) The columns of A are an orthonormal basis of $\mathbb{C}^n$.

   (iii) The rows of A are an orthonormal basis of $\mathbb{C}^n$.

2. Let $(V, \langle \cdot, \cdot \rangle)$ be a finite dimensional inner product space. Let S and T be two linear operators on V, and $\alpha \in \mathbb{F}$. Show the following.

   (a) $(S + T)^* = S^* + T^*$;

   (b) $(\alpha T)^* = \bar{\alpha} T^*$;

   (c) $(S \circ T)^* = T^* \circ S^*$;

3. Let $(V, \langle \cdot, \cdot \rangle)$ be a finite dimensional inner product space. Let $T : V \to V$ be a linear transformation. Show that the following are equivalent.

   (a) $T^*T = Id_V$;

   (b) $TT^* = Id_V$;

   (c) T sends every orthonormal basis to an orthonormal basis;

   (d) T sends some orthonormal basis to an orthonormal basis;

   (e) T preserves $\langle \cdot, \cdot \rangle$;

   (f) T preserves $|| \cdot ||$;

(g) T preserves distance;

4. We know (exercise 3 above) that a real square matrix A is orthogonal if and only if $AA^T = I$.

   (a) Show that if A is orthogonal then $\det(A) = \pm 1$.

   (b) Classify all 2×2 orthogonal matrices by showing for an orthogonal 2×2 real matrix A that

      (i) if $\det(A) = 1$ then the transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T_A(v) = Av$ is a rotation;

      (ii) if $\det(A) = -1$ then the transformation $T_A : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T_A(v) = Av$ is a reflection across a line L through the origin.