Math 491 - Linear Algebra II, Spring 2016

Homework 1 - Warm-up

Due: 2/2/16

Remark: Answers should be written in the following format:
A) Result.
B) If possible, the name of the method you used.
C) The computation or proof.

1. **The Change of Basis Matrix.** Let V be an n-dimensional vector space over a field $\mathbb{F}$.

   (a) Recall the definition of a basis of V and the coordinate map induced by a basis.

   (b) Suppose $\mathcal{B}$ and $\mathcal{C}$ are two bases for V. Show that there exists a unique matrix $M_{\mathcal{C},\mathcal{B}} \in M_n(\mathbb{F})$ such that, for every $v \in V$,

   $$[v]_{\mathcal{C}} = M_{\mathcal{C},\mathcal{B}}[v]_{\mathcal{B}}.$$

   Let $\mathcal{B} = \{v_1, \ldots, v_n\}$. Show that the matrix $M_{\mathcal{C},\mathcal{B}}$ has the formula

   $$M_{\mathcal{C},\mathcal{B}} = \begin{pmatrix} [v_1]_{\mathcal{C}} & \cdots & [v_n]_{\mathcal{C}} \end{pmatrix}.$$

   (c) Let $V = \mathbb{R}_{\leq 2}[x]$ be the vector space of polynomials of degree at most two with coefficients in $\mathbb{R}$. Let $\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{C} = \{1-x, 1+x, 1+x^2\}$ be two bases of V. Compute $M_{\mathcal{C},\mathcal{B}}$ and $M_{\mathcal{B},\mathcal{C}}$.

2. **The Matrix of a Linear Transformation.** Let V be an n-dimensional vector space over a field $\mathbb{F}$.

   (a) Recall the definition of a linear transformation $T : V \to V$.

   (b) Let $T : V \to V$ be a linear transformation. Show that there exists a unique matrix $[T]_{\mathcal{B}} \in M_n(\mathbb{F})$ such that for all $v \in V$

   $$[T(v)]_{\mathcal{B}} = [T]_{\mathcal{B}}[v]_{\mathcal{B}}.$$

Moreover, show that the matrix $[T]_\mathcal{B}$ has the formula

$$[T]_\mathcal{B} = \begin{pmatrix} [T(v_1)]_\mathcal{B} & \cdots & [T(v_n)]_\mathcal{B} \end{pmatrix}.$$

(c) Let $\mathcal{B}$ and $\mathcal{C}$ be two bases of V. Show that

$$[T]_\mathcal{C} = M_{\mathcal{C},\mathcal{B}}[T]_\mathcal{B} M_{\mathcal{B},\mathcal{C}}.$$

(d) Let $V = \mathbb{R}_{\leq 2}[x]$. Let $T : V \to V$ be given by the equation

$$T(p(x)) = (xp(x))'.$$

Compute $[T]_\mathcal{B}$ and $[T]_\mathcal{C}$ where $\mathcal{B} = \{1, x, x^2\}$ and $\mathcal{C} = \{1 - x, 1 + x, 1 + x^2\}$.

3. **Direct sum.** We say that a vector space V is a <u>direct sum</u> of the subspaces $V_1, \ldots, V_k \subset V$, and denote $V = V_1 \oplus \cdots \oplus V_k$, if every $v \in V$ can be written uniquely as $v = v_1 + \cdots + v_k$, where $v_i \in V_i$ for every $i = 1, \ldots, k$.

   (a) Suppose V is a finite dimensional vector space and $V_1, \ldots, V_k \subset V$ are subspaces. Show that the following are equivalent:

   1. $V = V_1 \oplus \cdots \oplus V_k$.
   2. For every collection of bases $\mathcal{B}_1$ for V_1, ..., $\mathcal{B}_k$ for V_k, their union $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_k$ is a basis for V.

   (b) (i) Let $V = F(\mathbb{R})$ be the vector space of functions on $\mathbb{R}$. Show that V is the direct sum of the subspaces generated by odd and even functions.

   (ii) Let $V = M_n(\mathbb{F})$ be the vector space of $n \times n$ matrices over $\mathbb{F}$. Show that V is the direct sum of the subspaces generated by symmetric and anti-symmetric matrices.

4. **Eigenvales and eigenvectors.** Let $T : V \to V$ be a linear transformation.

   (a) Recall the definition of eigenvectors and eigenvalues.

   (b) Show that if $v_1, \ldots, v_k \in V$ are eigenvectors of T associated with different eigenvalues $\lambda_1, \ldots, \lambda_k$, then they are linearly independent.

5. **Diagonalization** Let V be an n-dimensional vector space over $\mathbb{F}$. Let $T : V \to V$ be a linear transformation. Show that the following are equivalent:

   1. There exists a basis $\mathcal{B} = \{v_1, \ldots, v_n\}$ for V such that

$$[T]_\mathcal{B} = \begin{pmatrix} \lambda_1 & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \lambda_n \end{pmatrix}.$$

2. There exists $\mu_1, \ldots, \mu_k \in \mathbb{F}$ and subspaces $V_1, \ldots, V_k$ of V, such that $V = V_1 \oplus \cdots \oplus V_k$, and for each $i = 1, \ldots, k$ the action of T on V_i is given by multiplication by μ_i.

Remark
The lecturer and grader will be happy to help you with the homework. Please attend their office hours.