HW 9 Solutions

1 (a) $T(p(x)) = 2x(2x+3) + 2$

$$= \boxed{4x^2 + 6x + 2}$$

(b) 1. $x^2 + 3x + 2 = \alpha_1 \cdot 1 + \alpha_2 \cdot x + \alpha_3(2x^2+1)$

$$\begin{aligned} 2\alpha_3 &= 1 & \alpha_1 &= 3/2 \\ \alpha_2 &= 3 \longrightarrow & \alpha_2 &= 3 \\ \alpha_1 + \alpha_3 &= 2 & \alpha_3 &= 1/2 \end{aligned}$$

$$\boxed{[p]_{\mathcal{B}} = \begin{pmatrix} 3/2 \\ 3 \\ 1/2 \end{pmatrix}}$$

2. $T(p) = 4x^2 + 6x + 2 = \alpha_1 \cdot 1 + \alpha_2 \cdot x + \alpha_3(2x^2+1)$

$$\begin{aligned} 2\alpha_3 &= 4 & \alpha_1 &= 0 \\ \alpha_2 &= 6 \longrightarrow & \alpha_2 &= 6 \\ \alpha_1 + \alpha_3 &= 2 & \alpha_3 &= 2 \end{aligned}$$

$$\boxed{[T(p)]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 6 \\ 2 \end{pmatrix}}$$

3. $[T]_{\mathcal{B}} = \left([Tv_1]_{\mathcal{B}} \; [Tv_2]_{\mathcal{B}} \; [Tv_3]_{\mathcal{B}} \right)$

$Tv_1 = T(1) = 2x \cdot 0 + 0 = 0$ so $[Tv_1]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$Tv_2 = T(x) = 2x \cdot 1 + 0 = 2 \cdot x$ so $[Tv_2]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$

$Tv_3 = T(2x^2+1) = 2x(4x) + 4 = 4(2x^2+1)$ so $[Tv_3]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}$

$$\boxed{[T]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}}$$

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(b) 4. $[T]_{\mathcal{B}}[p]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 3/2 \\ 3 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ 2 \end{pmatrix} = [T(p)]_{\mathcal{B}}$.

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(a) Done!

(b) 1. $x^2 + 3x + 2 = \alpha_1 \cdot 1 + \alpha_2 \cdot (x+1) + \alpha_3 (x^2 + x + 1)$

$$\begin{aligned} \alpha_3 &= 1 \\ \alpha_3 + \alpha_2 &= 3 \\ \alpha_3 + \alpha_2 + \alpha_1 &= 2 \end{aligned} \longrightarrow \begin{aligned} \alpha_1 &= -1 \\ \alpha_2 &= 2 \\ \alpha_3 &= 1 \end{aligned}$$

$$\boxed{[p]_{\mathcal{C}} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}}$$

2. $[v_1]_{\mathcal{C}}$: $1 = \alpha_1 \cdot 1 + \alpha_2 \cdot (x+1) + \alpha_3 (x^2 + x + 1)$

$\longrightarrow \alpha_1 = 1, \ \alpha_2 = 0, \ \alpha_3 = 0$

$[v_1]_{\mathcal{C}} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$[v_2]_{\mathcal{C}}$: $x = \alpha_1 \cdot 1 + \alpha_2 (x+1) + \alpha_3 (x^2 + x + 1)$

$\longrightarrow \alpha_1 = -1, \ \alpha_2 = 1, \ \alpha_3 = 0$

$[v_2]_{\mathcal{C}} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$[v_3]_{\mathcal{C}}$: $2x^2 + 1 = \alpha_1 \cdot 1 + \alpha_2 (x+1) + \alpha_3 (x^2 + x + 1)$

$\longrightarrow \alpha_1 = 1 \quad \alpha_2 = -2 \quad \alpha_3 = 2$

$[v_3]_{\mathcal{C}} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

$$\boxed{M_{\mathcal{C} \leftarrow \mathcal{B}} = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{pmatrix}}$$

(2) (b) 3. $M_{\mathcal{C}\leftarrow B}[p]_B = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 3/2 \\ 3 \\ 1/2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} = [p]_{\mathcal{C}}$.

(c) 1. $[T]_{\mathcal{C}} = \left([Tw_1]_{\mathcal{C}} \quad [Tw_2]_{\mathcal{C}} \quad [Tw_3]_{\mathcal{C}} \right)$

$Tw_1 = T(1) = 0$ so $[Tw_1]_{\mathcal{C}} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$Tw_2 = T(x+1) = 2x \cdot 1 + 0 = 2x = \alpha_1 \cdot 1 + \alpha_2(x+1) + \alpha_3(x^2+x+1)$

$\longrightarrow \alpha_1 = -2 \quad \alpha_2 = 2 \quad \alpha_3 = 0$

so $[Tw_2]_{\mathcal{C}} = \begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}$

$Tw_3 = T(x^2+x+1) = 2x(2x+1) + 2$

$= 4x^2 + 2x + 2 = \alpha_1 \cdot 1 + \alpha_2 \cdot (x+1) + \alpha_3(x^2+x+1)$

$\longrightarrow \alpha_1 = 0 \quad \alpha_2 = -2 \quad \alpha_3 = 4$

so $[Tw_3]_{\mathcal{C}} = \begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix}$ and $\boxed{[T]_{\mathcal{C}} = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 4 \end{pmatrix}}$

2. $\left(\begin{array}{ccc|ccc} 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_2+R_3]{R_1-\frac{1}{2}R_3} \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & -1/2 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{array}\right)$

$\xrightarrow[R_1+R_2]{1/2\, R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1/2 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1/2 \end{array}\right)$

$$\boxed{(M_{\mathcal{C}\leftarrow B})^{-1} = \begin{pmatrix} 1 & 1 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1/2 \end{pmatrix}}$$

Note: $(M_{\mathcal{C}\leftarrow B})^{-1} = M_{B\leftarrow \mathcal{C}}$.

(2) (c) 3. $(M_{\mathcal{E}\leftarrow B})^{-1}[T]_{\mathcal{E}} M_{\mathcal{E}\leftarrow B} = \begin{pmatrix} 1 & 1 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 1/2 \end{pmatrix}\begin{pmatrix} 0 & -2 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{pmatrix}$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} = [T]_B$$

(3) (a) Note that $T - \lambda\,\mathrm{Id}$ is a linear transformation. Thus,

$$(T - \lambda \cdot \mathrm{Id})\, 0_V = 0_V = (T - \lambda \cdot \mathrm{Id})\, v$$

but

$$v \neq 0_V$$

so $T - \lambda \cdot \mathrm{Id}$ is not one-to-one. In particular, it is not invertible.

(b) $A_\lambda = [T - \lambda \cdot \mathrm{Id}]_B = \left([(T-\lambda \cdot \mathrm{Id})v_1]_B \;\; [(T-\lambda\cdot\mathrm{Id})v_2]_B \;\; [(T-\lambda\cdot\mathrm{Id})v_3]_B \right)$

$(T - \lambda \cdot \mathrm{Id})v_1 = T(1) - \lambda \cdot \mathrm{Id}(1) = 0 - \lambda = -\lambda$ so $[(T-\lambda\cdot\mathrm{Id})v_1] = \begin{pmatrix} -\lambda \\ 0 \\ 0 \end{pmatrix}$

$(T - \lambda \cdot \mathrm{Id})v_2 = T(x) - \lambda \cdot \mathrm{Id}(x) = 2x - \lambda x = (2-\lambda)x$ so $[(T-\lambda\,\mathrm{Id})v_2] = \begin{pmatrix} 0 \\ 2-\lambda \\ 0 \end{pmatrix}$

$(T - \lambda \cdot \mathrm{Id})v_3 = T(x^2) - \lambda \cdot \mathrm{Id}(x^2) = 4x^2 + 2 - \lambda x^2 = (4-\lambda)x^2 + 2$ so $[(T-\lambda\cdot\mathrm{Id})v_3] = \begin{pmatrix} 2 \\ 0 \\ 4-\lambda \end{pmatrix}$

and $$\boxed{A_\lambda = \begin{pmatrix} -\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{pmatrix}}$$

(3) (b) $P_A(\lambda) = \det A_\lambda = -\lambda(2-\lambda)(4-\lambda)$

since A_λ is upper triangular.

Clearly, the roots of $P_A(\lambda)$ are $\lambda = 0, 2, 4$.

(c) $\lambda = 0$ $\quad \ker A_0 = \ker \begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ $\quad 2z = 0$, $2y = 0$, $x = t$

$$V_0 = \ker A_0 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} t \;\middle|\; t \in \mathbb{R} \right\}$$

Then $\mathcal{B}_0 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ is a basis for V_0.

$\lambda = 2$ $\quad \ker A_2 = \ker \begin{pmatrix} -2 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ $\quad 2z = 0$, $-2x + 2z = 0$, $-2x = 0$ $\quad y = t$

$$V_2 = \ker A_2 = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} t \;\middle|\; t \in \mathbb{R} \right\}.$$

Then $\mathcal{B}_2 = \left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis for V_2.

$\lambda = 4$ $\quad \ker A_4 = \ker \begin{pmatrix} -4 & 0 & 2 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad -4x + 2z = 0$, $-2y = 0$, $x = \frac{1}{2}z$ $\quad z = t$

$$V_4 = \ker A_4 = \left\{ \begin{pmatrix} 1/2 \\ 0 \\ 1 \end{pmatrix} t \;\middle|\; t \in \mathbb{R} \right\}$$

Then $\mathcal{B}_4 = \left\{ \begin{pmatrix} 1/2 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for V_4.

(4) (a) $A_\lambda = A - \lambda \cdot I = \begin{pmatrix} 1-\lambda & 2 & 0 \\ 0 & 3-\lambda & 0 \\ 2 & -4 & 2-\lambda \end{pmatrix}$

$\det A_\lambda = (2-\lambda)(1-\lambda)(3-\lambda)$

has roots $\lambda = 1, 2, 3$

$\lambda = 1$ $\quad \ker A_1 = \ker \begin{pmatrix} 0 & 2 & 0 \\ 0 & 2 & 0 \\ 2 & -4 & 2 \end{pmatrix}$ $\quad 2y = 0 \Rightarrow y = 0$; $2x - 4y + 2z = 0$; $x = -z$ $\quad z = t$

$V_1 = \ker A_1 = \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} t \;\middle|\; t \in \mathbb{R} \right\}$

Then $\mathcal{B}_1 = \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for V_1.

$\lambda = 2$ $\quad \ker A_2 = \ker \begin{pmatrix} -1 & 2 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 0 \end{pmatrix}$ $\quad y = 0$; $-x + 2y = 0$; $x = 0$ $\quad z = t$

$V_2 = \ker A_2 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} t \;\middle|\; t \in \mathbb{R} \right\}$

Then $\mathcal{B}_2 = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for V_2.

$\lambda = 3$ $\quad \ker A_3 = \ker \begin{pmatrix} -2 & 2 & 0 \\ 0 & 0 & 0 \\ 2 & -4 & -1 \end{pmatrix}$ $\quad -2x + 2y = 0$; $2x - 4y - z = 0$; $y = t$ $\quad x = y$; $z = 4y - 2x$

$V_4 = \ker A_4 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} t \;\middle|\; t \in \mathbb{R} \right\}$

Then $\mathcal{B}_4 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$ is a basis for V_4.

(4) (b) $A_\lambda = A - \lambda \cdot I = \begin{pmatrix} 3-\lambda & 2 & 1 \\ 1 & 2-\lambda & -1 \\ -1 & 2 & 5-\lambda \end{pmatrix}$

$$\det A_\lambda = (3-\lambda)\Big((2-\lambda)(5-\lambda)+2\Big) - 2\Big(5-\lambda-1\Big) + 1\Big(2+(2-\lambda)\Big)$$

(cofactor expansion)

$$= (3-\lambda)\left(12 - 7\lambda + \lambda^2\right) - 10 + 2\lambda + 2 + 2 + 2 - \lambda$$

$$= 36 - 21\lambda + 3\lambda^2 - 12\lambda + 7\lambda^2 - \lambda^3 - 10 + 2\lambda + 6 - \lambda$$

$$= -\lambda^3 + 10\lambda^2 - 32\lambda + 32$$

$$= -(\lambda - 2)(\lambda - 4)^2$$

roots of $\lambda = 2, 4$.

$\lambda = 2$ ker $A_2 = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ -1 & 2 & 3 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ -1 & 2 & 3 & 0 \end{array}\right) \xrightarrow[R_2 - R_1]{R_3 + R_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 4 & 4 & 0 \end{array}\right) \xrightarrow[R_3 + 2R_2]{R_1 + R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x - z = 0$

$y = -z$

$z = t$

$$V_2 = \ker A_2 = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} t \;\middle|\; t \in \mathbb{R} \right\}$$

Then $\mathcal{B}_2 = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$ is a basis

for V_2.

4 (b) $\lambda = 4$ $\quad \ker V_4 = \ker \begin{pmatrix} -1 & 2 & 1 \\ 1 & -2 & -1 \\ -1 & 2 & 1 \end{pmatrix}$

$$\left(\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ 1 & -2 & -1 & 0 \\ -1 & 2 & 1 & 0 \end{array}\right) \xrightarrow[R_3 - R_1]{R_2 + R_1} \left(\begin{array}{ccc|c} -1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$-x + 2y + z = 0$

$y = s$

$z = t$ $\quad x = 2y + z$

$$V_4 = \ker A_4 = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} t \;\middle|\; s, t \in \mathbb{R} \right\}$$

Then a basis for V_4 is $\mathcal{B}_4 = \left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$.