Math 340 Spring 2014
HW8, due week of April 21-25
Bases

Remark. Answers should be written in the following format:
A) Result.
B) If possible the name of the method you used.
C) The computation.

Definitions. Let V be a vector space over $\mathbb{R}$.

- Define when a subset $S \subset V$ is <u>linearly independent</u>.
- Define when a subset $\mathcal{B} = \{v_{1,\ldots,}v_n\} \subset V$ is a <u>basis</u> for V.
- Let $v \in V$ and suppose $\mathcal{B} = \{v_{1,\ldots,}v_n\} \subset V$ is a basis for V. Define the <u>coordinate</u> $[v]_{\mathcal{B}} \in \mathbb{R}^n$, of v with respect to $\mathcal{B}$.
- Let $\mathcal{B} = \{v_{1,\ldots,}v_n\}$ be a basis for V. Let $T : V \to V$ be a linear transformation. Define $[T]_{\mathcal{B}} \in M_n(\mathbb{R})$, called the <u>matrix representing</u> T with respect to $\mathcal{B}$.
- Define what it means for V to have dimension n, denoted by $\dim(V) = n$.

1. **Basis and Coordinates**

   Let $V = \mathbb{R}^3$ and let

   $$v = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}.$$

   (a) Let

   $$u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, u_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

   Show that $\mathcal{B} = \{u_1, u_2, u_3\}$ is a basis for V.

   (b) Let

   $$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

   Show that $\mathcal{C} = \{v_1, v_2, v_3\}$ is a basis for V.

   (c) Compute the coordinate of v with respect to $\mathcal{B}$, i.e., $[v]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$, such that $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = v$.

(d) Compute the coordinate of v with respect to $\mathcal{C}$, i.e., $[v]_{\mathcal{C}} = \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix}$, such that $\beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3 = v$.

2. **The Matrix of a Linear Transformation**

We call a function $T : \mathbb{R}^n \to \mathbb{R}^n$ linear transformation if $T(u+v) = T(u) + T(v)$ and $T(\alpha u) = \alpha T(u)$, for all $u, v \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$.

Let

$$A = \begin{pmatrix} 7 & -3 & -3 \\ 10 & -4 & -7 \\ 0 & 0 & 3 \end{pmatrix}.$$

Consider the linear transformation $T_A : \mathbb{R}^3 \to \mathbb{R}^3$, given by $T_A(v) = Av$.

(a) Given a linear transformation T and a basis $\mathcal{B} = \{v_1, v_2, v_3\}$ we define the matrix of T with respect to $\mathcal{B}$ in the following way:

$$[T]_{\mathcal{B}} = ([T(v_1)]_{\mathcal{B}}, [T(v_2)]_{\mathcal{B}}, [T(v_3)]_{\mathcal{B}}).$$

So to compute the i^{th} column of the matrix we apply T to the i^{th} basis vector, and express the result in coordinates with respect to the basis $\mathcal{B}$. Let $\mathcal{E} = \{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$. Compute $[T_A]_{\mathcal{E}}$.

(b) Let

$$u_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}, u_3 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

Let $\mathcal{C} = \{u_1, u_2, u_3\}$. Note that that $\mathcal{C}$ is a basis for $\mathbb{R}^3$. Compute $D = [T_A]_{\mathcal{C}}$. Now let $C = (u_1, u_2, u_3)$. Note that C is a matrix, where as $\mathcal{C}$ is a basis. Show that $C^{-1}AC = D$.

3. **Basis and Dimension**

For the following subspaces, W, find a basis and compute the dimension of the subspace.

(a) The system

$$\begin{cases} x + 2y + 4z = 0 \\ 3x - 5y + z = 0 \\ x + 3y + 5z = 0 \end{cases}$$

and $W \subset \mathbb{R}^3$ is the set of solutions to the system above.

(b) The matrix

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 5 & 1 & -4 \\ -3 & -1 & 2 \end{pmatrix}$$

and $W = Ker(A) \subset \mathbb{R}^3$.

(c) Let $V = \{\text{Polynomials of degree at most 3}\}$. Let $W = \{p \in V |\ p'' = 0\}$, where p'' denotes the second derivative of p.

(d) The matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix}$$

and $W_3 = \{v \in R^3 | (A - 3I)v = 0\}$. $W_2 = \{v \in R^3 | (A - 2I)v = 0\}$.

Remarks

- You are very much encouraged to work with other students. However, submit your work alone.
- The TA and the Lecturer will be happy to help you with the homework. Please visit the office hours.

Good Luck!