Math 340 - Solutions - Homework #8

1) a) $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

So $\mathrm{Det}\left(\begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix}\right) = -2 \neq 0$. So this matrix is one-to-one meaning the set is linearly independent, and onto meaning it spans $\mathbb{R}^3$. So $\mathcal{B}$ is a basis.

b) $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. $\mathrm{Det}\left(\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}\right) = 1 \neq 0$. So $\mathcal{C}$ is a basis.

c) $\left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 1 & 0 & 1 & 4 \\ 0 & 1 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & -1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 2 & 2 \end{array}\right)$

$\to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$

d) $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right)$

$$[v]_{\mathcal{B}} = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \qquad [v]_{\mathcal{C}} = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$$

2) $A = \begin{pmatrix} 7 & -3 & -3 \\ 10 & -4 & -7 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix}$

a) $[T_A(e_i)]_I = [a_i]_I$ So $[T_A]_I = A$.

b) $T_A(u_1) = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ $T_A(u_2) = \begin{pmatrix} 6 \\ 10 \\ 0 \end{pmatrix}$ $T_A(u_3) = \begin{pmatrix} 0 \\ -3 \\ 3 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 6 & 0 \\ 2 & 5 & -1 & 2 & 10 & -3 \\ 0 & 0 & 1 & 0 & 0 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 6 & 0 \\ 0 & -1 & -1 & 0 & -2 & -3 \\ 0 & 0 & 1 & 0 & 0 & 3 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 6 & 0 \\ 0 & -1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 3 \end{array}\right)$$

I was solving 3 linear equations at once, as we do in matrix inversion

$[T_A(u_1)]_e = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $[T_A(u_2)]_C = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$ $[T_A(u_3)]_e = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$

<u>Computing C^{-1}</u>

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 2 & 5 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -5 & 3 & 3 \\ 0 & 1 & 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$D = [T_A]_e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

$$C^{-1}AC = \begin{pmatrix} -5 & 3 & 3 \\ 2 & -1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 7 & -3 & -3 \\ 10 & -4 & -7 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 \\ 2 & 5 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -5 & 3 & 3 \\ 4 & -2 & -2 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 \\ 2 & 5 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

3) a) $\begin{pmatrix} 1 & 2 & 4 \\ 3 & -5 & 1 \\ 1 & 3 & 5 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 4 \\ 0 & -11 & -11 \\ 0 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

$\longrightarrow \begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x_1 = -2x_3$, $x_2 = -x_3$, $x_3 = x_3$

Solutions $= \left\{ t \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \middle| \, t \in \mathbb{R} \right\}$ so $\mathcal{B} = \left\{ \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \right\}$ is a basis.

b) $\begin{pmatrix} 2 & 3 & 1 \\ 5 & 1 & -4 \\ -3 & -1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 2 & 3 & 1 \\ 1 & -5 & -6 \\ -3 & -1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -5 & -6 \\ 2 & 3 & 1 \\ -3 & -1 & 2 \end{pmatrix}$

$\longrightarrow \begin{pmatrix} 1 & -5 & -6 \\ 0 & 13 & 13 \\ 0 & -16 & -16 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -5 & -6 \\ 0 & 1 & 1 \\ 0 & -16 & -16 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$x_1 = x_3$
$x_2 = -x_3$
$x_3 = x_3$

$\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\}$ is a basis

c) W is all vectors $p(x) = ax^2 + bx + c$ s.t. $p''(x) = 2a = 0$. So $a = 0$. So W is all vectors $p(x) = bx + c$. Therefore $\mathcal{B} = \{1, x\}$ is a basis

d) $W_2 = \ker \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 3 \end{pmatrix} = \left\{ \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in V \middle| \, v_3 = 0 \right\}$ so $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a basis

$W_3 = \ker \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$. $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$x_1 = 0$
$x_2 = -x_3$
$x_3 = x_3$

$\mathcal{B} = \left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}$ is a basis.