Math 340 Spring 2014
HW7 - due in discussion the week of April 7-11
Vector Spaces

Remark. Answers should be written in the following format:
A) Result.
B) If possible the name of the method you used.
C) The computation.

Definitions.

- What property defines the zero vector 0_V in a vector space V?
- What property defines the additive inverse v' of a vector v in a vector space V?
- What does it mean for a subset W of a vector space V to be a <u>subspace</u>?
- Let V be a vector space. Define what it means for a map $T : V \to V$, to be a <u>linear transformation</u>.
- For a linear transformation $T : V \to V$, define the <u>kernel</u> of T, denoted $\ker(T)$, and the <u>image</u> of T, denoted $\mathrm{Im}(T)$.
- Define a <u>linear combination</u> of vectors $v_1, ..., v_k$ in a vector space V.
- Define the <u>span</u>, denoted $Span(X)$ of a subset X of vectors in a vector space V.

1. **Vector Space Definition**

   Decide which of the following sets V, with addition $+$ and scalar multiplication $\cdot$, and $0_V \in V$ are vector spaces?

   (a) The set V of 2×2 matrices with the following addition and multiplication.:

   $$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix},$$

   $$\alpha \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}, \ \alpha \in \mathbb{R}.$$

   With zero element as

   $$0_V = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

   (b) The trace of a matrix A, denoted $tr(A)$, is the sum of the entries along the diagonal. Define $V = \{A \in M_2(\mathbb{R}) | \ tr(A) = 0\}$ with addition and scalar multiplication and zero 0_V as in part a,

(c) The set V of solutions to a homogeneous system of m equations in n unknowns (Since such solutions are elements of $\mathbb{R}^n$ addition and scalar multiplication will be defined as they are in $\mathbb{R}^n$).

(d) The set V of vectors in $\mathbb{R}^n$ with all integer coordinates, with addition and scalar multiplication as defined in $\mathbb{R}^n$.

2. **Subspaces**

Let V be a vector space. A subset $W \subset V$ is a subspace of V if W is also a vector space using the same addition and scalar multiplication in V, and $0_V \in W$.

All that one needs to do in order to check whether or not W is indeed a subspace, is to show that $0_V \in W$, and that if we add two arbitrary vectors in W, the sum is in W, and that if we multiply an arbitrary vector in W by a scalar, the product is in W.

(a) Let V be a vector space. Let $T : V \to V$ be a linear transformation, i.e., T is a map from V to V that satisfies $T(u + v) = T(u) + T(v)$, and $T(\alpha \cdot v) = \alpha \cdot T(v)$ for every $u, v \in V$, $\alpha \in \mathbb{R}$. Show that $Ker(T) = \{v \in V \text{ such that } T(v) = 0_V\}$, and $Im(T) = \{T(v) \mid v \in V\}$ are subspaces of V.

(b) Let T be as above. Let $\lambda \in \mathbb{R}$. Let $W = \{\mathbf{v} \in V | T(\mathbf{v}) = \lambda \cdot \mathbf{v}\}$. Is W a subspace of V? Vectors in W are called eigenvectors with eigenvalue λ.

(c) Let $F(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R}\}$ be the set of function from the reals to the reals. Let $W = \{f \in F(\mathbb{R})| \ f(0) = 0\}$. Is W a subspace of $F(\mathbb{R})$?

(d) Let $W = \{f \in F(\mathbb{R})| \ f(0) = 1\}$. Is W a subspace of $F(\mathbb{R})$?

(e) Let $W = \{0_V\}$. Is W a subspace of $V = \mathbb{R}^n$?

3. **Linear Combinations and Span**

(a) Let V be a vector space. Recall that a linear combination of the vectors $v_1, v_2, \ldots, v_k \in V$ is any vector of the from $u = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_k v_k$, where $\alpha_1, ..., \alpha_k \in \mathbb{R}$.
Let

$$u = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, v = \begin{pmatrix} 4 \\ -6 \\ 2 \end{pmatrix}, w = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} \in \mathbb{R}^3.$$

Is w a linear combination of u and v? In other words, does there exist $\alpha, \beta \in \mathbb{R}$ such that $w = \alpha u + \beta v$?

(b) Let X be a subset of a vector space V. We define

$$Span(X) = \{\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k | \ \ x_i \in X, \ \alpha_i \in \mathbb{R}\}.$$

In other words, the span of a set of vectors is the set of all linear combinations of vectors in the set. Verify that for any non-empty $X \subset V$, we have $Span(X)$ is a subspace of V.

(c) Using u, v, and w from above. Is w in $Span(\{u, v\})$?

(d) For the following sets of vectors give a brief geometric description of their spans.

1. Let $u = \begin{pmatrix} 0 \\ 1 \\ -7 \end{pmatrix}$. Describe $Span(\{u\})$.
2. Let $u = \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}, v = \begin{pmatrix} -2 \\ 2 \\ 8 \end{pmatrix}$. Describe $Span(\{u, v\})$.
3. Let $u = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, v = \begin{pmatrix} 1 \\ 7 \\ 5 \end{pmatrix}$ $w = \begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix}$. Describe $Span(\{u, v, w\})$.

Remarks

- You are very much encouraged to work with other students. However, submit your work alone.
- The TA and the Lecturer will be happy to help you with the homework. Please visit the office hours.

Good Luck!