Math 340 - Solutions HW - # 7

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1) a) · Associativity: $\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix}\right] + \begin{pmatrix} i & j \\ k & l \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} + \begin{pmatrix} i & j \\ k & l \end{pmatrix}$

$$= \begin{pmatrix} (a+e)+i & (b+f)+j \\ (c+g)+k & (d+h)+l \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \left[\begin{pmatrix} e & f \\ g & h \end{pmatrix} + \begin{pmatrix} i & j \\ k & l \end{pmatrix}\right] = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e+i & f+j \\ g+k & h+l \end{pmatrix}$$

$$= \begin{pmatrix} a+(e+i) & b+(f+j) \\ c+(g+k) & d+(h+l) \end{pmatrix}$$

Associativity in V follows by associativity in $\mathbb{R}$.

· Commutativity also follows by commutativity in $\mathbb{R}$

· $\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

· $\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

· $\alpha\left[\beta\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] = \alpha\begin{pmatrix} \beta a & \beta b \\ \beta c & \beta d \end{pmatrix} = \begin{pmatrix} \alpha\beta a & \alpha\beta b \\ \alpha\beta c & \alpha\beta d \end{pmatrix} = (\alpha\beta)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

· $1\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

· $\alpha\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \beta\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix} + \begin{pmatrix} \beta a & \beta b \\ \beta c & \beta d \end{pmatrix}$

$$= \begin{pmatrix} \alpha a+\beta a & \alpha b+\beta b \\ \alpha c+\beta c & \alpha d+\beta d \end{pmatrix} = \begin{pmatrix} (\alpha+\beta)a & (\alpha+\beta)b \\ (\alpha+\beta)c & (\alpha+\beta)d \end{pmatrix} = (\alpha+\beta)\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

· $\alpha\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix}\right] = \alpha\begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} = \begin{pmatrix} \alpha(a+e) & \alpha(b+f) \\ \alpha(c+g) & \alpha(d+h) \end{pmatrix}$

$$= \begin{pmatrix} \alpha a+\alpha e & \alpha b+\alpha f \\ \alpha c+\alpha g & \alpha d+\alpha h \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix} + \begin{pmatrix} \alpha e & \alpha f \\ \alpha g & \alpha h \end{pmatrix} = \alpha\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \alpha\begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

V is a vector space

b) * Since this vector space is a subset of the one in part a, we don't need to check the axioms, only closure of the operations.

<u>Addition</u> Let $u, v \in V$

$$\underset{u}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} + \underset{v}{\begin{pmatrix} e & f \\ g & h \end{pmatrix}} = \underset{u+v}{\begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}}$$

If $a+d=0$ and $e+h=0$ then $a+e+d+h=0$. So $u+v \in V$

<u>Multiplication</u> Let $u \in V$ $\alpha \in \mathbb{R}$

$$\alpha u = \alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$$

If $a+d=0$ then $\alpha a + \alpha d = 0$. So $\alpha u \in V$.

V is a vector space.

c) $V \leq \mathbb{R}^n$ so we will only check closure. See note * above.

<u>Addition</u> Suppose $x, y \in V$. Let A represent the $m \times n$ homogeneous system. So $Ax = Ay = 0$. So $Ax + Ay = A(x+y) = 0$. So $x+y \in V$.

<u>Multiplication</u> Suppose $x \in V$ $\alpha \in \mathbb{R}$. So $Ax = 0$. Then $A(\alpha x) = \alpha A(x) = \alpha \cdot 0 = 0$. So $\alpha x \in V$.

V is a vector space

d) V is not a vectorspace. $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \in V$ but $(\frac{1}{2})e_1 = \begin{pmatrix} 1/2 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ is not.

2) a) • $\ker(T)$ is a subspace. If $u, v \in \ker(T)$, $T(u) = T(v) = 0$. So $T(u+v) = T(u) + T(v) = 0 + 0 = 0$. So $u+v \in \ker(T)$. If $\alpha \in \mathbb{R}$ then $T(\alpha u) = \alpha T(u) = \alpha \cdot 0 = 0$. So $\alpha u \in \ker(T)$.

• $\mathrm{im}(T)$ is a subspace. If $u, v \in \mathrm{Im}(T)$ $\exists x, y \in V$ s.t. $T(x) = u$, $T(y) = v$. So $T(x+y) = T(x) + T(y) = u + v$. So $u+v \in \mathrm{Im}(T)$. $T(\alpha x) = \alpha T(x) = \alpha u$ so $\alpha u \in \mathrm{Im}(T)$.

b) Suppose $u, v \in W$. Then $T(u) = \lambda u$ and $T(v) = \lambda v$. So $T(u+v)$ $= T(u) + T(v) = \lambda u + \lambda v = \lambda(u+v)$. Let $\alpha \in \mathbb{R}$. $T(\alpha u) = \alpha T(u)$ $= \alpha \lambda u = \lambda(\alpha u)$. So $u+v, \alpha u \in W$. W is a subspace.

c) Let $f, g \in W$, $\alpha \in \mathbb{R}$. Then $f(0) = g(0) = 0$. $(f+g)(0) = f(0) + g(0)$ $= 0 + 0 = 0$. $(\alpha f)(0) = \alpha f(0) = \alpha \cdot 0 = 0$. So $f+g, \alpha f \in W$.

d) Let $f(x) = 1$. Then $f \in W$. $\frac{1}{2} \in \mathbb{R}$. $(\frac{1}{2} f)(x) = \frac{1}{2} f(x) = \frac{1}{2}$. $(\frac{1}{2} f)(0) = \frac{1}{2} \neq 1$. So $(\frac{1}{2} f) \notin W$. So W is not a subspace.

e) W is a subspace. Let $u, v \in W$, $\alpha \in \mathbb{R}$. Then $u = v = 0$. So $u+v = 0$. Also $\alpha u = 0$. So $u+v, \alpha u \in W$.

3) a)

$$u = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} \quad v = \begin{pmatrix} 4 \\ -6 \\ 2 \end{pmatrix} \quad w = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 1 & 4 & 5 \\ 3 & -6 & 6 \\ 2 & 2 & 7 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 4 & 5 \\ 0 & -18 & -9 \\ 0 & -6 & -3 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 4 & 5 \\ 0 & 1 & 1/2 \\ 0 & -6 & -3 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 4 & 5 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{array}\right) \quad \text{So } 3u + \tfrac{1}{2} v = w.$$

b) Suppose $u, v \in \mathrm{Span}(X)$. Then $u = \alpha_1 x_1 + \cdots + \alpha_n x_n$ and $v = \beta_1 x_1 + \cdots + \beta_n x_n$. $u+v = (\alpha_1 + \beta_1) x_1 + \cdots + (\alpha_n + \beta_n) x_n$ and $\gamma u = (\gamma \alpha_1) x_1 + \cdots + (\gamma \alpha_n x_n)$ for any $\gamma \in \mathbb{R}$. So $u+v, \gamma u \in \mathrm{Span}(X)$. So $\mathrm{Span}(X)$ is a subspace.

c) Yes we showed this in part a.

d) 1) A line through the origin.

2) A line through the origin since $v = 2u$.

3) All of $\mathbb{R}^3$, since the three vectors are linearly independent. This can be verified by row reduction.