Math 340 - Solutions - HW #5

1) a) Suppose A is invertible. Let $S = T_{(A^{-1})}$. Let $v \in \mathbb{R}^n$.

$T_A \circ S(v) = T_A(A^{-1}v) = AA^{-1}v = Iv = v$

$S \circ T_A(v) = S(Av) = A^{-1}Av = Iv = v$

So $S \circ T_A = T_A \circ S = Id$ the identity transformation.

b) Suppose T_A is invertible. Let

$$B = \begin{pmatrix} (T_A)^{-1}(e_1) & \dots & (T_A)^{-1}(e_n) \end{pmatrix}$$

Then $(T_A)^{-1} = T_B$. So $T_{AB} = T_A \circ T_B = T_I$. Therefore $AB = I$. $T_{BA} = T_B \circ T_A = T_I$. So $BA = I$. So $B = A^{-1}$.

c) Suppose $0 \neq v \in \ker(A)$. If A were invertible then $A^{-1}Av = A^{-1}0 = 0$. So $v = 0$. But we know it doesn't. So A can not be invertible.

2) a)
$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & 3 & -1 \\ 0 & 1 & -2 & 1 \end{array}\right)$$

$$\left(\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 4 & -3 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 2 & 0 & 3 & -1 \\ 0 & -1 & -2 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{cc|cc} 1 & 0 & 3/2 & -1/2 \\ 0 & 1 & 2 & -1 \end{array}\right) \qquad A^{-1} = \begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} \qquad B^{-1} = \begin{pmatrix} 3/2 & -1/2 \\ 2 & -1 \end{pmatrix}$$

$$C = B^{-1}A^{-1} = \begin{pmatrix} 3/2 & -1/2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 11/2 & -2 \\ 8 & -3 \end{pmatrix}$$

$$ABC = \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 4 & -3 \end{pmatrix}\begin{pmatrix} 11/2 & -2 \\ 8 & -3 \end{pmatrix} = \begin{pmatrix} 6 & -4 \\ 16 & -11 \end{pmatrix}\begin{pmatrix} 11/2 & -2 \\ 8 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

b) Suppose A, B are invertible. Let $C = B^{-1}A^{-1}$. Then $(AB)C = ABB^{-1}A^{-1}$
$= AA^{-1} = I$. $C(AB) = B^{-1}A^{-1}AB = B^{-1}B = I$. So $C = (AB)^{-1}$.

c) Fact: Let $T: \mathbb{R}^n \to \mathbb{R}^n$. Then T is one-to-one if and only if T is onto.

Suppose A, B are $n \times n$ matrices and AB is invertible. If B is not invertible it is neither one-to-one nor onto. So $\exists\, x, y$ s.t. $x \neq y$ and $Bx = By$. So $ABx = ABy$. Then AB is not invertible, but we know it is. By contradiction B must be invertible. Suppose A is not invertible. Then A is neither one-to-one nor onto. So $\exists\, v \in \mathbb{R}^n$ s.t. $\forall x \in \mathbb{R}^n$ $v \neq Ax$. So $\forall x \in \mathbb{R}^n$ $v \neq ABx$. So AB is not onto. Then AB is not invertible, but we know it is. So by contradiction A must be invertible.

3) a) $$\left(\begin{array}{cc|cc} -2 & 4 & 1 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 1 & -2 & -1/2 & 0 \\ 3 & 5 & 0 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 1 & -2 & -1/2 & 0 \\ 0 & 11 & 3/2 & 1 \end{array}\right)$$

$$\longrightarrow \left(\begin{array}{cc|cc} 1 & -2 & -1/2 & 0 \\ 0 & 1 & 3/22 & 1/11 \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 1 & 0 & -5/22 & 4/22 \\ 0 & 1 & 3/22 & 2/22 \end{array}\right)$$

$$A^{-1} = \frac{1}{22}\begin{pmatrix} -5 & 4 \\ 3 & 2 \end{pmatrix}$$

b) $$\left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 4 & 6 & -2 & 0 & 1 & 0 \\ 1 & 3 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 2 & -4 & 1 & 0 \\ 0 & 2 & 2 & -1 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 2 & -4 & 1 & 0 \\ 0 & 0 & 0 & 3 & -1 & 1 \end{array}\right)$$ B is not invertible.

4) When inverting an $n \times n$ invertible matrix by row-reduction, one row reduces an $n \times 2n$ matrix, until the i^{th} column is e_i. Multiplying a row by a scalar, and subtracting one row from another both take $2n$ operations, on the order of n. In order to make one column e_i one must execute each of these operations on the order of n times, so each column will require n^2 operations. We want to reduce n columns to e_i resulting in a total of approximately n^3 operations.