Math 340 - Solutions - HW #4

1) a) $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ $e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$T_A(e_1) = Ae_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = e_1$$

$$T_A(e_2) = Ae_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = 2e_2$$

$$T_A(e_3) = Ae_3 = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} = 3e_3$$

b) $B = \begin{pmatrix} 3/2 & -1/2 & 1/2 \\ -1 & 2 & 1 \\ -1/2 & 1/2 & 5/2 \end{pmatrix}$ $f_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $f_2 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ $f_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$$T_B(f_1) = Bf_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = f_1$$

$$T_B(f_2) = Bf_2 = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} = 2f_2$$

$$T_B(f_3) = Bf_3 = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} = 3f_3$$

c) $$V = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 1 & 0 & 1 & b \\ 0 & 1 & 1 & c \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & -1 & 1 & b-a \\ 0 & 1 & 1 & c \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & -1 & a-b \\ 0 & 0 & 2 & c+b-a \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & -1 & a-b \\ 0 & 0 & 1 & (c+b-a)/2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & a \\ 0 & 1 & 0 & (a+c-b)/2 \\ 0 & 0 & 1 & (c+b-a)/2 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 0 & 0 & (a+b-c)/2 \\ 0 & 1 & 0 & (a+c-b)/2 \\ 0 & 0 & 1 & (c+b-a)/2 \end{array}\right)$$

d) $u = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ $\quad T_B(u) = Bu = \begin{pmatrix} -3/2 - 1 + 1/2 \\ 1 + 4 + 1 \\ 1/2 + 1 + 5/2 \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \\ 4 \end{pmatrix}$

$U = \alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3$ where $\alpha_1 = 0$, $\alpha_2 = -1$, $\alpha_3 = 2$

$$T_B(u) = T_B(-f_2 + 2f_3) = -T_B(f_2) + 2T_B(f_3) = -\begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 6 \\ 4 \end{pmatrix}$$

2) a) $B = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$ $\quad \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & -2 & -1 & 1 \end{array}\right)$

$$\rightarrow \left(\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 0 & 1 & 1/2 & -1/2 \end{array}\right) \rightarrow \left(\begin{array}{cc|cc} 1 & 0 & -1/2 & 3/2 \\ 0 & 1 & 1/2 & -1/2 \end{array}\right) \quad B^{-1} = \begin{pmatrix} -1/2 & 3/2 \\ 1/2 & -1/2 \end{pmatrix}$$

b) $A = \begin{pmatrix} 5/2 & 3/2 \\ -1/2 & 1/2 \end{pmatrix}$ $\quad D = BAB^{-1} = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} -1/2 & 3/2 \\ 1/2 & -1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

c) $A = B^{-1}DB$. $\quad A^{13} = \underbrace{(B^{-1}D\cancel{B})(\cancel{B^{-1}}D\cancel{B}) \cdots \cancel{(B^{-1}}D\cancel{B})(\cancel{B^{-1}}DB)}_{13 \text{ times}}$

$= B^{-1}D^{13}B$

Since $D^{13} = \begin{pmatrix} 1 & 0 \\ 0 & 2^{13} \end{pmatrix}$

$$A = \begin{pmatrix} -1/2 & 3/2 \\ 1/2 & -1/2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2^{13} \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1/2 & 3/2 \\ 1/2 & -1/2 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 2^{13} & 2^{13} \end{pmatrix}$$

$$= \begin{pmatrix} 3 \cdot 2^{12} - 1/2 & 3 \cdot 2^{12} - 3/2 \\ -2^{12} + 1/2 & -2^{12} + 3/2 \end{pmatrix}$$