Math 340 Spring 2014
Homework #3, Due in the week of Feb. 10-14, 2014
Algebra of Matrices and Vectors

Remark. Answers (beside for question 4) should be written in the following format:
A) Result.
B) If possible the name of the method you used.
C) The computation.

1. **Matrices act on vectors**

   (a) Let

   $$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

   Compute $R_\theta(\mathbf{v})$ for $\theta = 0, \frac{2\pi}{3}, \frac{4\pi}{3}, 2\pi$.

   (b) Let

   $$R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

   The matrix R acts on vectors in $\mathbb{R}^2$ by rotating them by 90 degrees counterclockwise. The matrix S acts on vectors in $\mathbb{R}^2$ by reflecting them across the diagonal line $y = x$. Compute RS and SR. Verify that $RS \neq SR$

2. **Linear combinations**

   We say that $\mathbf{u} \in \mathbb{R}^n$ is a linear combination of $\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n} \in \mathbb{R}^n$ if there exist real numbers $x_1, x_2, \ldots, x_n \in \mathbb{R}$ such that

   $$\mathbf{u} = x_1\mathbf{v_1} + x_2\mathbf{v_2} + \cdots + x_n\mathbf{v_n}.$$

   Let

   $$A = \begin{pmatrix} 1 & 0 & -2 \\ 1 & 3 & 1 \\ 2 & 0 & 1 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix}.$$

   (a) Show that if $A\mathbf{x} = \mathbf{b}$ then $\mathbf{b}$ is a linear combination of the column vectors that make up A.

   (b) Find real numbers x_1, x_2, x_3 such that $\mathbf{b} = x_1\mathbf{v_1} + x_2\mathbf{v_2} + x_3\mathbf{v_3}$, where $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are the first second and third columns of A, respectively.

3. **Matrices and inverses**

(a) Let

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}.$$

1. Show that $AB = BA = I$ where I is the 2×2 identity matrix (1 on the diagonal, 0 everywhere else).
2. Show that $A\mathbf{x} = \mathbf{b}$.
3. Show that if $A\mathbf{y} = \mathbf{b}$, then $\mathbf{y} = \mathbf{x}$.

(b) Let

$$C = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}.$$

Show that there is no matrix D, such that $CD = I$.

4. Let $A, B, C, D \in M_n(\mathbb{R})$.

(a) Define what it means for the matrix B to be the <u>inverse</u> of A.

(b) Define what it means for the matrices C and D to <u>commute</u>.

Remarks

- You are very much encouraged to work with other students. However, submit your work alone.
- The TA and the Lecturer will be happy to help you with the homework. Please visit the office hours.

Good Luck!