Math 340 Spring 2014
HW10 due week of May 5-9 2014
Inner Product Spaces, Symmetric Matrices and their Diagonalization

Remark. Answers should be written in the following format:
A) Result.
B) If possible the name of the method you used.
C) The computation.

1. **Definition of Inner Product Space.** Let V be a vector space. An <u>inner product</u> on V is a binary function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ such that,

   - Symmetry: $\langle u, v\rangle = \langle v, u\rangle$;
   - Additivity: $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$;
   - Homogeneity: $\langle au, v\rangle = a \cdot \langle u, v\rangle$;
   - Positivity: $\langle v, v\rangle \geq 0$;
   - Nondegenerate: $\langle v, v\rangle = 0$ if and only if $v = 0$,

   for every $u, v, w \in V$, $a \in \mathbb{R}$. We call the pair $(V, \langle \cdot, \cdot \rangle)$ an <u>inner product space</u>. Don't be confused by notation, this is just saying that V is a space with an inner product.

   (a) Check that the following are inner product spaces.

      1. $V = \mathbb{R}^n$ and $\langle u, v\rangle = x_1y_1 + x_2y_2 + \cdots + x_ny_n$. Here x_i and y_i are the coordinates of u and v.
      2. $V = \{\text{Polynomials of degree at most } 2\}$ and $\langle p, q\rangle = \int_0^1 p(t)q(t)dt$, for every $p, q \in V$.

   (b) Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product. The <u>norm</u> is a function $|| \cdot || : V \to \mathbb{R}$ defined as
   $$||v|| = \sqrt{\langle v, v\rangle}.$$
   For each of the following vectors v find the norm by using the inner products from Part a.

      1. The vector
      $$v = \begin{pmatrix} 1 \\ -2 \\ 2 \\ -5 \end{pmatrix}.$$
      2. The polynomial $p(t) = t^2 - 2t + 2$.

2. **Orthogonality.** Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. Let $u, v \in V$. We say that u and v are <u>orthogonal</u>, written $u \perp v$, if $\langle u, v\rangle = 0$.

(a) Prove the Pythagorean Theorem. If $u \perp v$ then, $||u||^2 + ||v||^2 = ||u+v||^2$.

(b) Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. An orthonormal basis $\mathcal{B} \subset V$, is a basis satisfying two properties: **(1)** If $u, v \in \mathcal{B}$ and $u \neq v$, then $u \perp v$; **(2)** If $u \in \mathcal{B}$, then $||u|| = 1$. Check that the following bases are orthonormal:

1. The space $V = \mathbb{R}^3$ with the inner product from 1.a. and $\mathcal{B} = \{e_1, e_2, e_3\}$, the standard basis.
2. The space $V = \{\text{Polynomials of degree at most } 2\}$ with the inner product from 1.a. and $\mathcal{B} = \{1, \sqrt{12}(x - 1/2), \sqrt{180}(x^2 - x + 1/6)\}$.
3. The space $V = \{A \in M_2(\mathbb{R}) |\ tr(A) = 0\}$ with the following inner product, if

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, B = \begin{pmatrix} e & f \\ g & h \end{pmatrix},$$

then

$$\langle A, B \rangle = a \cdot e + b \cdot f + c \cdot g + d \cdot h.$$

Consider

$$\mathcal{B} = \{ \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & -1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \}.$$

(c) Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space and $W \subset V$ be a subspace of V. We define the orthogonal complement of W to be the subspace $W^\perp = \{v \in V |\ v \perp w\ \forall w \in W\}$. For the following subspaces W, find a basis for $W^\perp$.

1. $W = Span(\{u_1, u_2)\} \subset \mathbb{R}^4$, where

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}.$$

2. $W = \{A \in M_2(\mathbb{R}) |\ tr(A) = 0\}$.

(d) Explain why if $S = \{v_1, v_2, \ldots, v_n\} \subset V$ is a set of nonzero orthogonal vectors, then S is a linearly independent set.

(e) The following algorithm (the **Gram-Schmidt** procedure) produces an orthonormal basis for any finite dimensional inner product space $(V, \langle \cdot, \cdot \rangle)$.

Start: Let $S = \varnothing$ (the empty set).

Step 1: Find a nonzero vector $v \in V$ which is perpendicular to every vector in S, i.e., $v \in Span(S)^\perp$. If no such v exists, go to **Done**.

Step 2: Normalize v (divide v by $\|v\|$, so its length is now $= 1$).

Step 3: Add this vector to S and go to **Step 1.**

Done: Let $\mathcal{B} = \mathcal{S}$. This is an orthonormal basis for V.

Apply the algorithm to $V = \{v \in \mathbb{R}^4 |\ x_1 + x_2 + x_3 + x_4 = 0\}$ where x_i are the coordinates of v.

3. **Diagonalization of Symmetric Matrices**

A symmetric matrix A, is a matrix such that $A = A^t$, where A^t is the transpose of A. In other words the entry in row i and column j, is equal to the entry in row j and column i. One fact about symmetric matrices is that, if we consider $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ with the inner product from question 1.a., then for any $u, v \in \mathbb{R}^n$, $\langle Au, v \rangle = \langle u, Av \rangle$. Symmetric matrices also have the very nice property that they are diagonalizable, and, moreover, every symmetric matrix A has an **orthonormal basis of eigenvectors**. The last fact is true using Gram–Schmidt, and the fact that eigenvectors of different eigenvalues are orthogonal. This can be seen by the following calculation. Suppose u is an eigenvector of A with eigenvalue λ, and v is an eigenvector of A with eigenvalue μ, e.g. $Au = \lambda u$ and $Av = \mu v$, where $\lambda \neq \mu$. Then,

$$\lambda \cdot \langle u, v \rangle = \langle Au, v \rangle = \langle u, Av \rangle = \mu \cdot \langle u, v \rangle.$$

So $(\mu - \lambda) \cdot \langle u, v \rangle = 0$, therefore $\langle u, v \rangle = 0$.

(a) Find an orthonormal basis of eigenvectors for

$$A = \begin{pmatrix} 3 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 6 \end{pmatrix}.$$

This matrix will have three distinct eigenvalues. Due to arguments made in the opening paragraph of this question, the corresponding eigenvectors will be orthogonal. Therefore in order to find an orthonormal basis, one must only divide each of these vectors by their norm.

(b) Find an orthonormal basis of eigenvectors for

$$A = \begin{pmatrix} -1 & -3 & 0 \\ -3 & -1 & 0 \\ 0 & 0 & -4 \end{pmatrix}.$$

In this instance you will find two roots of the characteristic polynomial but one root will have multiplicity two. So there will be one eigenspace of dimension 1 and one eigenspace of dimension 2. If you find an orthonormal basis for each of these, using Gram–Schmidt, their union will be an orthonormal basis of eigenvectors of A.

4. **Definitions.**

(a) Define what it means for a pair $(V, \langle \cdot, \cdot \rangle)$ to be an inner product space.

(b) Let $(V, \langle \cdot, \cdot \rangle)$ be inner product space.

1. Define what it means for two vectors $u, v \in V$ to be orthogonal.
2. Define what it means for a basis $\mathcal{B} \subset V$ to be orthonormal.
3. For a subset $S \subset V$ define the orthogonal complement $S^{\perp}$.

(c) Define what it means for a square matrix to be symmetric.

Remarks

- You are very much encouraged to work with other students. However, submit your work alone.
- The TA and the Lecturer will be happy to help you with the homework. Please visit the office hours.

Good Luck!